Project Questions for

EEE588: Design of Multivariable Control Systems

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Motivation. This project is intended to cover topics which are essential for multiple-input multiple-output (MIMO) control system analysis and design.

Problem 1 (Nonlinear Car Dynamics and Linearization) *Consider a car of mass m with horizontal force u due to the engine and aerodynamic force βv^2 ($\beta > 0$).*
(a) Derive the nonlinear differential equation which relates the velocity v to the input force u. Answer: $\dot{v} = -\frac{\beta}{m}v^2 + \frac{1}{m}u$.
(b) What equilibrium velocity v_e does the constant force u_e result in? Answer: $v_e = \sqrt{\frac{u_e}{\beta}}$.
(c) Linearize the model obtained in (a) about the equilibrium (u_e, v_e) computed in (b). Answer: $\delta\dot{v} = -\frac{2\beta v_e}{m}\delta v + \frac{1}{m}\delta u$ *where* $\delta v \stackrel{\text{def}}{=} v - v_e$ *and* $\delta u \stackrel{\text{def}}{=} u - u_e$.
Note: In practice, the δ qualifiers are often dropped. While this may be confusing to a "beginner," it must be emphasized that a linear model which came from a nonlinear model (by assumption) considers only small perturbations about an equilibrium condition.
(d) For the model obtained in (c), what is the transfer function from the force input (perturbation) δu to the car velocity (perturbation) δv? What is the associated pole? Is the system stable, unstable, or marginally stable?
(e) Provide a state-space representation for the above car model.
(f) Suppose that the vehicle has mass $m = 1$ and $\beta = 0.5$. We would like to design a cruise control system for our vehicle to operate near $v_e = 1$ (Obviously these are rigged numbers.). Specifically, we want to design a controller K and a reference command pre-filter W such that the resulting closed loop system satisfies the following clodsed loop design specifications:

1. *closed loop system is stable;*
2. *closed loop system exhibits zero steady state error to step commands, step disturbances at the plant output, and step disturbances at the plant input;*
3. *closed loop system exhibits a settling time t_s to a unit step reference command of approximately $t_s \approx 5$ seconds;*
4. *closed loop system exhibits an overshoot M_p to a unit step reference command of approximately $M_p \approx 4.3\%$.*

Design Procedure: Use the standard negative feedback system we have been considering - with controller K and reference command pre-filter W (i.e. use Figure 1 with $H = 1$. First design the controller K. You will need an integrator to satisfy (2). You will need a zero to satisfy (1) and (2) - think root locus folks. What do zeros do the root locus? (3) and (4) tell you where your closed loop poles should be - real part and damping factor ζ, respectively. You need to recall basic knowledge about second order systems. To satisfy (4) you will need a pre-filter - once again, you need to recall basic knowledge about second order systems. Without a pre-filter, the closed loop system will exhibit undesirably large overshoot due to the zero in the compensator K.

(g) How does your design change if $m = 2000$, $\beta = 20$, and $v_e = 50$?

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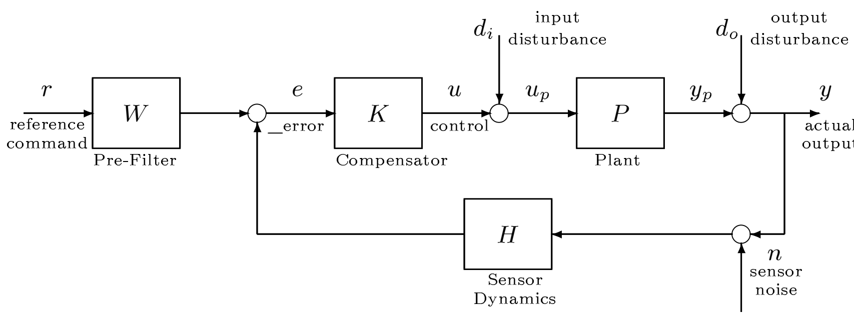

Figure 1: Standard Negative Feedback System with Pre-Filter

Comment 1 (Aerodynamic Force on a Vehicle)

It is useful to note that the aerodynamic force on a vehicle is proportional to qA where $q \stackrel{\text{def}}{=} \frac{1}{2}\rho v^2$ is the dynamic pressure, ρ is the density of air, A is the cross-sectional area of the vehicle, and v the speed. It thus follows that β is proportional to $\frac{1}{2}\rho A v$.

To help one relate the above exercise to reality, it is useful to note that while Formula 1 race cars typically have a lower β than do city busses, the big fat city bus typically has a lower $\frac{\beta}{m}$ ratio. Does this make sense to you? Which one will take longer to come to rest when given an initial speed? ■

Problem 2 (Matrix Mapping of a Unit Circle.)

In this problem we examine how a 2×2 matrix maps the unit cicle. For each of the following matrices compute a singular value decomposition (using MATLAB) and show how the matrix maps the unit circle in the x_1x_2-plane via $y = Hx$ to the y_1y_2-plane. For each matrix plot the unit circle and the associated ellipse. Show all important singular vector information on your plots.

$$(a) \quad H = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix} \qquad (1)$$

$$(b) \quad H = \begin{bmatrix} 5 & 5 \\ 0 & 1 \end{bmatrix} \qquad (2)$$

$$(c) \quad H = \begin{bmatrix} 5 & -5 \\ 0 & 1 \end{bmatrix} \tag{3}$$

$$(d) \quad H = \begin{bmatrix} 5 & 0 \\ 5 & 1 \end{bmatrix} \tag{4}$$

$$(e) \quad H = \begin{bmatrix} 5 & 0 \\ -5 & 1 \end{bmatrix} \tag{5}$$

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Problem 3 (Change of Units: Poles, Transfer Function Matrix, Transmission Zeros)
This problem is intended to show that a change of units does not alter the fundamental characteristics of a MIMO LTI system.

In many applications, it is desirable to

- *change the units of the variables being used or*
- *change the coordinate system being used.*

In either case, a transformation of variables *is required. Given this, suppose that we are given an LTI system with state space representation* (A_1, B_1, C_1, D_1)*:*

$$\dot{x}_1 = A_1 x_1 + B_1 u_1 \tag{6}$$

$$y_1 = C_1 x_1 + D_1 u_1. \tag{7}$$

Suppose that we desire to transform the variables u_1, x_1, y_1*. This can be accomplished via the following linear transformations:*

$$u_2 = S_u u_1 \tag{8}$$

$$x_2 = S_x x_1 \tag{9}$$

$$y_2 = S_y y_1 \tag{10}$$

where $S_u \in \mathcal{R}^{m\times m}$, $S_x \in \mathcal{R}^{n\times n}$, $S_y \in \mathcal{R}^{p\times p}$ *are invertible (nonsingular) matrices. Given this, the new state space representation* (A_2, B_2, C_2, D_2) *is given by:*

$$\dot{x}_2 = A_2 x_2 + B_2 u_2 \tag{11}$$

$$y_2 = C_2 x_2 + D_2 u_2 \tag{12}$$

where

$$A_2 = S_x A_1 S_x^{-1} \quad B_2 = S_x B_1 S_u^{-1} \tag{13}$$

$$C_2 = S_y C_1 S_x^{-1} \quad D_2 = S_y D_1 S_u^{-1}. \tag{14}$$

The transformation matrix S_x *is often called a* similarity transformation *and the matrices* A_1 *and* A_2 *are said to be* similar *to one another. As might be expected, it can be shown that changing units does not change the fundamental properties of a system.*

Consider the two state space representations: (A_1, B_1, C_1, D_1) *for system* S_1 *and* (A_2, B_2, C_2, D_2) *for system* S_2*. It is assumed that the state space representations are related by equations 13-14. Let*

H_1 and H_2 denote the respective transfer function matrices of the systems.

(a) How do the poles of system S_2 differ from those of system S_1?
Answer: They are identical. Show it!
(b) How do the eigenvectors of system S_2 differ from those of system S_1?
Answer: $x_2 = S_x x_1$.
(c) Relate the system transfer function matirix H_2 to H_1.
Answer: $H_2 = S_y H_1 S_u^{-1}$.
(d) How do the transmission zeros of system S_2 differ from those of system S_1?
Answer: They are identical. Show it!
(e) How do the transmission zeros directions of system S_2 differ from those of system S_1?
Answer: $u_2 = S_u u_1$, $x_2 = S_x x_1$. ■

Comment 3 (Changing Units and Invariance)
The above shows that changing units does not alter pole locations and transmission zero locations. The associated directionality information (e.g. eigenvectors, zero directions, etc.) does change! Scaling is very important. It is a MIMO issue - not a SISO issue. Why? Because for MIMO systems, the vectors you deal with often contain variables with differnt units (e.g. degrees, ft, ft/sec, etc.). If you use "screwy" units for a MIMO system, then the directional information will look "screwy." You want to use the "right" units - that is, units that allow you to properly compare the numbers in your vectors; units that allow you to properly compare "apples" and "oranges." Determining the "right" units, however, is an art. How does one compare apples and oranges or degrees and feet/second? How many apples is one orange equivalent to? I suppose it depends on ones taste as well as good-old supply and demand. A typical rule of thumb - sometimes called Bryson's rule - is to scale all of the variables according to their anticipated maximum deviations from a nominal value. ■

Problem 4 (MIMO Frequency Response Singular Value Analysis)
The purpose of this problem is to illustrate how the singular value decomposition may be used to study the steady state behavior of MIMO LTI dynamical systems driven by sinusoidal inputs.

Let H denote an $m \times xm$ transfer function matrix for a MIMO LTI system. Assuming zero initial conditions, the output vector y is related to the input vector z by an (s-domain) equation of the form:

$$y(s) = H(s)z(s). \tag{15}$$

Suppose that

$$H(j\omega) = U\Sigma V^H \tag{16}$$

where

$$U_{kl} = u_{kl}e^{j\theta_{kl}} \tag{17}$$

is the kl^{th} entry of U,

$$\Sigma = diag(\sigma_1, \ldots, \sigma_m) \tag{18}$$

σ_i is the i^{th} singular value of $H(j\omega)$, and

$$V_{kl} = v_{kl}e^{jphi_{kl}} \tag{19}$$

is the kl^{th} entry of V.

Note that the l^{th} column of V is the right singular vector of $H(j\omega)$ which corresponds to the l^{th} singular value σ_l of $H(j\omega)$. Similarly, note that the l^{th} column of U is the left singular vector of $H(j\omega)$ which corresponds to the l^{th} singular value σ_l of $H(j\omega)$.

Suppose further that the system H is stable and that H is being excited by a vector input u consisting of sinusoids of frequency ω. The stability assumption on H guarantees that all transients will decay and that the steady state output vector y exists and will also consist of sinusoidal signals having frequency ω.

(a) Suppose that the k^{th} input is chosen to be

$$z_k = v_{kl}\ \cos(\omega t + \phi_{kl}). \tag{20}$$

Note that it has been constructed using information from the l^{th} column of V; i.e. using information from the l^{th} right singular vector of $H(j\omega)$.

Given the above, show that the steady state output in the k^{th} output channel is given by

$$y_k = \sigma_l\ u_{kl}\ \cos(\omega t + \theta_{kl}). \tag{21}$$

Note that this output has been constructed using information from the l^{th} column of U; i.e. using information from the l^{th} left singular vector of $H(j\omega)$.

Hint: What steady state output does $v_k\ e^{j\omega t}$ produce? Call it w. What steady state output does $\overline{v_k}\ e^{-j\omega t}$ produce? Call it $\overline{w}$. Recall Euler's formula: $w + \overline{w} = 2Rew = 2|w|\cos(\angle w)$. This holds for any complex scalar quantity w. Express each entry of $z = \frac{1}{2}\left[v_k\ e^{j\omega t} + \overline{v_k}\ e^{-j\omega t}\right]$ as a real cosine. What steady state output does z produce?

(b) The following shows how the above result is used in practice. Suppose that

$$\begin{aligned} H(j10) &= U\Sigma V^H & (22)\\ &= \begin{bmatrix} 0.9501 + 0.8913i & 0.6068 - 0.4565i \\ -0.2311 - 0.7621i & 0.4860 + 0.0185i \end{bmatrix} & (23) \end{aligned}$$

where

$$\begin{aligned} U &= \begin{bmatrix} 0.6654 + 0.6461j & 0.1999 + 0.3160j \\ -0.1333 - 0.3494j & 0.1187 + 0.9198j \end{bmatrix} & (24)\\ &= \begin{bmatrix} 0.9275\ e^{j44.1583\,\text{deg}} & 0.3739\ e^{j57.6869\,\text{deg}} \\ 0.3739\ e^{-j110.8821\,\text{deg}} & 0.9275\ e^{j82.6465\,\text{deg}} \end{bmatrix} & (25)\\ \Sigma &= \begin{bmatrix} 1.5956 & 0.0000 \\ 0.0000 & 0.7736 \end{bmatrix} & (26)\\ V &= \begin{bmatrix} 0.9433 & -0.3320 \\ 0.0235 + 0.3312j & 0.0669 + 0.9409j \end{bmatrix} & (27)\\ &= \begin{bmatrix} 0.9433\ e^{j0.0000\,\text{deg}} & 0.3320\ e^{j180.0000\,\text{deg}} \\ 0.3320\ e^{j85.9345\,\text{deg}} & 0.9433\ e^{j85.9345\,\text{deg}} \end{bmatrix} & (28) \end{aligned}$$

Maximally Amplified Input. *Given the above singular value decomposition for $H(j10)$, it follows that if the input z is selected as follows*

$$z(t) = \begin{bmatrix} 0.9433\ \cos(10t + 0.0000\,\text{deg}) \\ 0.3320\ \cos(10t + 85.9345\,\text{deg}) \end{bmatrix} \tag{29}$$

i.e. "in the direction of the right singular vector corresponding to the maximum singular value," then the steady state output will be "in the direction of the left singular vector associated with the maximum singular value" and given by

$$y(t) \quad = \quad 1.5956 \begin{bmatrix} 0.9275 \ \cos(10t + 44.1583\,\text{deg}) \\ 0.3739 \ \cos(10t - 110.8821\,\text{deg})\,\text{deg}) \end{bmatrix}. \tag{30}$$

Minimally Amplified Input. *Similarly, if the input z is selected as follows*

$$z(t) \quad = \quad \begin{bmatrix} 0.3320 \ \cos(10t + 180.0000\,\text{deg}) \\ 0.9433 \ \cos(10t + 85.9345\,\text{deg}) \end{bmatrix} \tag{31}$$

i.e. "in the direction of the right singular vector corresponding to the minimum singular value," then the steady state output will be "in the direction of the left singular vector associated with the minimum singular value" and given by

$$y(t) \quad = \quad 0.7736 \begin{bmatrix} 0.3739 \ \cos(10t + 57.6869\,\text{deg}) \\ 0.9275 \ \cos(10t + 82.6465\,\text{deg}) \end{bmatrix}. \tag{32}$$

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Problem 5 (Modified F8 Aircraft Analysis)
In this problem we examine the longitudinal dynamics of a modified F8 aircraft. The F8 is an "old-fashioned" aircraft used by NASA as part of their "fly-by-wire" research program. We have purposely modified the longitudinal dynamics by adding an oversized flaperon on the wing. This flaperon does not exist on the real F8. Such surfaces do exit on other aircraft to provide precision maneuvering (e.g. X-29).

Model Discussion. *It will be assumed that the aircraft is flying in the vertical plane with its wings level (i.e. without banking or turning) so that we can study its motion in the vertical plane, i.e. its longitudinal dynamics.*

The important variables that characterize the aircraft's motion are:

- *horizontal velocity v of the aircraft,*
- pitch angle θ *(angle between body and horizontal inertial x-axis),*
- *pitch rate $q = \dot{\theta}$,*
- angle of attack α *(angle between body and the velocity vector).*

It is the angle of attack *(of the wing) which is responsible for producing lift. As the aircraft moves through the air, drag is also produced. Both lift and drag are roughly proportional to α for small α. The* flight path angle γ *is defined as follows*

$$\gamma \stackrel{\text{def}}{=} \theta - \alpha \tag{33}$$

It is the angle between the velocity vector and the horizontal inertial x-axis.

Controls. *The longitudinal motion of the aircraft is controlled by two hinged aerodynamic control surfaces:*

- *the elevator δ_e is located on the horizontal tail,*
- *the flaperons δ_f are located on the wings. These are like ailerons, except they move in the same direction - not differentially. As such, we view the flaperons as one control.*

Deflection of either of these surfaces (elevator or flaperons) downward causes the airflow to be deflected downward. This produces a force that induces a nose-down moment about the cg. As the aircraft rotates it changes its angle of attack which results in additional forces and moments (see dynamics below).

Engine Thrust. *The longitudinal motion of the aircraft is also influenced by the thrust generated by the engines. However, in this problem we shall fix the thrust to be a constant and we shall not use it as a dynamic control variable. (Actually, the dynamic coordination of the thrust, elevator, and flaperon becomes important and significant when the aircraft is in its landing configuration.)*

Nonlinear Model and Linearization. *In general, the differential equations that model the aircraft longitudinal motion are nonlinear. These nonlinear differential equations can be linearized about a particular equilibrium (steady state) flight condition which is characterized by constant airspeed, cg location, trimmed angle of attack, trimmed pitch angle (so that $\gamma = 0$, i.e. level flight), and trimmed elevator deflection δ_e to maintain zero pitch rate ($q = 0$).*

One can then obtain a system of linear ordinary differential equations with constant coeficients that describe the deviations of the relavent quantities from their constant equilibrium (trimmed) values. The resulting system yields a linear time invariant (LTI) multiple-input multiple-output (MIMO) - in our case two-input two-output (TITO) - dynamical system.

Equilibrium Conditions. *The F8 model which follows is based on a linearization about the following equilibrium flight condition:*

- *Altitude: 20,000 ft = 6095 meters*
- *Speed Mach: 0.9 = 281.58 m./sec = 916.6 ft/sec*
- *Dynamic Pressure: 550 lbs/sq ft = 26429 N/sq m*
- *Trim Pitch Angle: 2.25 deg*
- *Trim Angle of Attack: 2.25 deg*
- *Trim Elevator Attack: -2.65 deg*

Unscaled Longitudinal Dynamics. *Given the above, the unscaled F8 longitudinal dynamics are as follows:*

$$\dot{x}_1 = A_1 x_1 + B_1 u_1 \tag{34}$$

$$y_1 = C_1 x_1 \tag{35}$$

where

$$A_1 = \begin{bmatrix} -0.8 & -0.0006 & -12 & 0 \\ 0 & -0.014 & -16.64 & -32.2 \\ 1 & -0.0001 & -1.5 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{36}$$

$$B_1 = \begin{bmatrix} -19 & -2.5 \\ -0.66 & -0.5 \\ -0.16 & -0.6 \\ 0 & 0 \end{bmatrix} \tag{37}$$

$$C_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \tag{38}$$

$$u_1 = \begin{bmatrix} \delta_e & \text{elevator deflection from trim (rad)} \\ \delta_f & \text{flaperon deflection (rad)} \end{bmatrix} \tag{39}$$

$$x_1 = \begin{bmatrix} q & \text{pitch rate (rad/sec)} \\ v & \text{perturbation from equilibrium horizontal speed } (ft/sec) \\ \alpha & \text{perturbed angle of attack from trim (rad)} \\ \theta & \text{perturbed pitch angle from trim (rad)} \end{bmatrix} \tag{40}$$

$$y_1 = \begin{bmatrix} \theta & \text{perturbed pitch angle from trim (rad)} \\ \gamma & \text{perturbed flight path angle from trim (rad)} \end{bmatrix} \tag{41}$$

System Poles and Transmission Zeros.
(a) What are the system poles? For each pole, what is the associated time constant, settling time, damping factor, and undamped natural frequency associated with each pole?
Terminology: The low frequency poles are referred to by aero engineers as the phugoid mode. *The high frequency poles are referred to by aero engineers as the* short period mode. *These modes are discussed in any text on aircraft flight dynamics (e.g. Etkin, Blakelock, etc.). You may also want to examine Tae-young Kim's MS thesis, March 2000, ASU. She describes these modes for a commercial aircraft.*
(b) What are the transmission zeros?

Controllability and Observability.
(c) Is the system controllable?
(d) Is the system observable?

Multivariable Frequency Response.
(e) Plot the singular values of the system transfer function matrix versus frequency. Relate the peaks, valleys, and breaks to the poles and zeros above (just as you would on a Bode plot).

Changing Units: Nonsingular Coordinate Transformations. *Radians is not an appropriate unit to use. It is too large. Degrees is much more appropriate. Wouldn't you agree? Suppose, therefore, that we scale the system (and rearrange the states* [1]*) as follows:*

$$u = S_u u_1 \tag{42}$$
$$x = S_x x_1 \tag{43}$$
$$y = S_y y_1 \tag{44}$$

where

$$S_u = \begin{bmatrix} r2d & 0 \\ 0 & r2d \end{bmatrix} \tag{45}$$

[1] The reason for rearranging the states will be seen when we study LQ servos.

$$S_x = \begin{bmatrix} 0 & 0 & 0 & r2d \\ 0 & 0 & -r2d & r2d \\ r2d & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{46}$$

$$S_y = \begin{bmatrix} r2d & 0 \\ 0 & r2d \end{bmatrix} \tag{47}$$

and $r2d = 180/\pi$ *is a radian to degree conversion factor. The new units (and variables) are then as follows:*

$$u = \begin{bmatrix} \delta_e & \text{elevator deflection from trim (deg)} \\ \delta_f & \text{flaperon deflection (deg)} \end{bmatrix} \tag{48}$$

$$x = \begin{bmatrix} \theta & \text{perturbed pitch angle from trim (deg)} \\ \gamma & \text{perturbed flight path angle from trim (deg)} \\ q & \text{pitch rate (deg/sec)} \\ v & \text{perturbation from equilibrium horizontal speed } (ft/sec) \end{bmatrix} \tag{49}$$

$$y = \begin{bmatrix} \theta & \text{perturbed pitch angle from trim (deg)} \\ \gamma & \text{perturbed flight path angle from trim (deg)} \end{bmatrix} \tag{50}$$

Characteristics of Scaled System.
(c) What is the new state space representation A, B, C*?*
(d) How is the new transfer function matrix $H(s) = C(sI-A)^{-1}B + D$ *related to the original transfer function matrix* $H_1(s) = C_1(sI - A_1)^{-1}B_1 + D_1$*?*
(e) How have the poles changed? Prove that they should be the same. Hint: Show that

$$det(sI - A) = det(sI - A_1). \tag{51}$$

Have the eigenvectors changed? How are the new eigenvectors related to the old eigenvectors?
(f) How have the transmission zeros changed? Prove that they should be the same. Hint: It is a fact that

$$det \begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix} = det(sI-A)det(C(sI-A)^{-1}B + D) \tag{52}$$

Use this fact to show that

$$det \begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix} = det S_y \, det \begin{bmatrix} sI - A_1 & -B_1 \\ C_1 & D_1 \end{bmatrix} det S_u^{-1}. \tag{53}$$

(g) Have the zero (input and state) directions changed? How are the new direction vectors related to the old direction vectors?
(h) How is the new controllability matrix $\mathcal{C}$ *related to the old controllability matrix* $\mathcal{C}_1$*? Show that* $\mathcal{C}$ *has no left null space if and only if* $\mathcal{C}_1$ *has no left null space. That is, the controllability properties of a system are invariant under a nonsingular coordinate transformation.*
(i) How is the new observability matrix $\mathcal{O}$ *related to the old observability matrix* $\mathcal{O}_1$*? Show that* $\mathcal{O}$ *has no right null space if and only if* $\mathcal{O}_1$ *has no right null space. That is, the observability properties of a system are invariant under a nonsingular coordinate transformation.*

DC Analysis.
(j) Compute the dc gain matrix H(0)? What does this matrix tell us? Suppose that we would like the system outputs to be $\theta = 1$ *and* $\gamma = 0$. *What steady state controls are required to support this flight condition? Suppose that we would like the system outputs to be* $\theta = 0$ *and* $\gamma = 1$. *What steady state controls are required to support this flight condition?*

SVD Analysis at DC.
(k) Do a singular value decomposition for the dc gain matrix $H(0)$. *Show how* $H(0)$ *maps the unit circle in the* $x_1 x_2$*-plane to the* $y_1 y_2$*-plane. Explain which input and output directions are associated with* σ_{max}. *Explain which input and output directions are associated with* σ_{min}. *Carefully label your plots in each plane - carefully indicating all singular vector information.*

SVD Analysis at Phugoid Frequency.
(l) Do a singular value decomposition for $H(j\omega_{ph})$ *where* $\omega_{ph} = 0.0270$ *rad/sec is the phugoid mode's damped natural frequency. Express your svd in polar form. Explain which input and output directions are associated with* σ_{max}. *Explain which input and output directions are associated with* σ_{min}. *Show how the the above svd information (in polar form) can be used to construct a sinusoidal input that will be maximally amplified. What is the corresponding output sinusoid? Show how the the above svd information (in polar form) can be used to construct a sinusoidal input that will be minimally amplified. What is the corresponding output sinusoid?*

SVD Analysis at Short Period Frequency.
(m) Do a singular value decomposition for $H(j\omega_{sp})$ *where* $\omega_{sp} = 3.6336$ *rad/sec is the short period mode's damped natural frequency. Express your svd in polar form. Explain which input and output directions are associated with* σ_{max}. *Explain which input and output directions are associated with* σ_{min}. *Show how the the above svd information (in polar form) can be used to construct a sinusoidal input that will be maximally amplified. What is the corresponding output sinusoid? Show how the the above svd information (in polar form) can be used to construct a sinusoidal input that will be minimally amplified. What is the corresponding output sinusoid?*

Multivariable Frequency Response.
(n) Plot the singular values of the new system transfer function matrix $H(s)$ *versus frequency. Relate the peaks, valleys, and breaks to the poles and zeros above (just as you would on a Bode plot). How does the new singular value plot differ from the old singular value plot?*

Modal Analysis.
(o) Let x_{ph} *denote a right eigenvector corresponding to the phugoid mode (*$\lambda_{ph} = -0.0058 + j0.0264$*). Let* x_{sp} *denote a right eigenvector corresponding to the short period mode (*$\lambda_{sp} = -1.1512 + j3.4464$*).*

- *What is* $x(t)$ *when* $x(0) = Re\{x_{ph}\}$*? Give an analytic expression for* $x(t)$. *Use MATLAB to determine* $x(t)$. *Discuss the phugoid mode. What can be said about the size of the angle of attack relative to the other variables? What if you had used* $x(0) = Im\{x_{ph}\}$*?*

  Partial Observations: This lightly damped mode takes very long to decay. The aircraft climbs and drops - exchanging kinetic and potential energy. Angle of attack (AOA) α *doesn't vary much. This implies that the flight path angle is approximately equal to the pitch angle (i.e.* $\gamma \approx \theta$*). Speed variations* v *are significant.*

- *What is* $x(t)$ *when* $x(0) = Re\{x_{sp}\}$*? Give an analytic expression for* $x(t)$. *Use MATLAB to* *determine* $x(t)$*. What can be said about the size of the speed relative to the other variables? What if you had used* $x(0) = Im\{x_{sp}\}$*?*

Partial Observations: This lightly damped mode decays very quickly. The aircraft's speed v *doesn't vary much. Pitch rate variations* q *are significant.*

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Problem 6 (State Space Arithmetic)

Suppose a system P *has a state space representation* (A_p, B_p, C_p, D_p) *with input* u_p*, state* x_p*, and output* y_p*. Suppose a system* K *has a state space representation* (A_k, B_k, C_k, D_k) *with input* u_k*, state* x_k*, and output* y_k*.*

Determine a state space representation for the following transfer function matrices:

(a) $P + K$*.*

(b) PK*.*

(c) KP*.*

(d) $T = PK[I + PK]^{-1}$*. Hint: Consider the transfer function matrix from reference commands to output for a unity feedback system with plant* P *and controller* K*.*

(e) $S = [I + PK]^{-1}$*. Hint: Consider the transfer function matrix from reference commands to error for a unity feedback system with plant* P *and controller* K*.*

(f) $KS = K[I + PK]^{-1}$*. Hint: Consider the transfer function matrix from reference commands to controls for a unity feedback system with plant* P *and controller* K*.*

(g) $PS = P[I + PK]^{-1}$*. Hint: Consider the transfer function matrix from disturbances at plant input to output for a unity feedback system with plant* P *and controller* K*.*

Hint: Sketch a block diagram for each case and propagate the signals around the block diagram. ■

Problem 7 (Controllability and Observability)

Consider the system S_1*:*

$$\dot{x}_1 = u_1 \qquad y_1 = -x_1 + u_1 \tag{54}$$

and the system S_2*:*

$$\dot{x}_2 = x_2 + u_2 \qquad y_2 = x_2. \tag{55}$$

(a) What is the transfer function of the system S_1*?* S_2*?*

(b) Discuss the controllability and observability properties of the cascade systems S_1S_2 $(u = u_2, u_1 = y_2, y = y_1)$ *and* S_2S_1 $(u = u_1, u_2 = y_1, y = y_2)$*, respectively. Also discuss the stability of the cascade systems (Careful!).*

Note: Use the controllabiltiy and observability matrix rank tests to support your discussion. Also use the PBH eigenvalue-eigenvector tests to support your discussion. Finally, how do your conclusions relate to pole-zero cancellations at the "input" and "output"?

(c) Suppose that a unity negative feedback loop is wrapped around the cascade system S_1S_2*. What can be said about the controllability and observability properties of the closed loop system? Is the closed loop system stable? (Careful!)*

(d) Repeat (c) for the cascade system S_2S_1.

■

Problem 8 (General Solution, Minimum Norm Solution)
Consider the linear system:

$$Ax = b$$

with $A = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$ *and* $b = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$.

(a) Parameterize the set of all solutions $x \in \mathcal{R}^2$.
(b) What is the minimum norm (Euclidean two norm) solution $x = [x_1 \ \ x_2]^T$.

■

Problem 9 (Bases, Fundamental Spaces, Least Squares, Minimum Norm Solution)
Consider the linear system:

$$Ax = b$$

with $A = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$ *and* $b = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$.

(a) Obtain a basis for each of the four (4) fundamental subspaces: $\mathcal{R}(A)$, $\mathcal{R}(A^T)$, $\mathcal{N}(A)$, $\mathcal{N}(A^T)$.
(b) Parameterize the set of all $x \in \mathcal{R}^2$ which minimize $\|b - Ax\|_2$.
(c) What is the minimum norm $x \in \mathcal{R}^2$ which minimizes $\|b - Ax\|_2$.

■

Problem 10 (Reachability, Control Construction)
Consider the cascade system in Figure 2. Suppose that system 1 has a state space representation

$$\dot{x}_1 = -x_1 + u \tag{56}$$
$$v = -x_1 + u \tag{57}$$

and system 2 has a state space representation

$$\dot{x}_2 = v \tag{58}$$
$$y = x_2 \tag{59}$$

(a) Find a state space representation (A, B, C, D) for the cascade system.
(b) Determine an eigenvalue-eigenvector decomposition for A. Show that $A = \lambda_1 x_1 y_1^H + \lambda_2 x_2 y_2^H$ where the $\{x_i\}_{i=1}^2$ and $\{y_i^H\}_{i=1}^2$ are right and left eigenvectors of A, respectively. Answer: $\lambda_1 = 0$, $\lambda_2 = -1$, $x_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $y_1^H = [\,-1 \quad 1\,]$, $y_2^H = [\,1 \quad 0\,]$.
(c)Using the eigenvalue-eigenvector decomposition determined in (b), compute e^{At}. Show that $e^{At} = e^{\lambda_1 t} x_1 y_1^H + e^{\lambda_2 t} x_2 y_2^H$. Compute $e^{At}B$. Show that $e^{At}B = e^{\lambda_1 t} x_1 y_1^H B + e^{\lambda_2 t} x_2 y_2^H B = e^{\lambda_2 t} x_2 y_2^H B$. Why does this last equality hold?

(d) Assuming zero initial conditions, compute $x(t) = \int_0^t e^{A(t-\tau)} Bu(\tau)d\tau$*. Show that*

$$x(t) = \left[\int_0^t e^{\lambda_2(t-\tau)} y_2^H Bu(\tau) d\tau \right] x_2. \tag{60}$$

(e) Does there exist a control u *such that* $x(1) = x_2$*? If so, find one. Is the control you found unique? If not, find another.*
(f) Show that there does not exist a control u *such that* $x(t_f) = x_1$*. We say that* x_1 *is not* reachable. *Find a control* u *that minimizes the Euclidean norm (distance)* $\|x_1 - x(1)\|_2$*. Hint: Let* $k = \left[\int_0^{t_f} e^{\lambda_2(t_f-\tau)} y_2^H Bu(\tau) d\tau \right]$ *and minimize* $\|x_1 - x_2 k\|_2$ *with respect to* k*. Determine a control* u *for the* k *you computed. Is your control unique? If not, find another control that minimizes the above norm.*

■

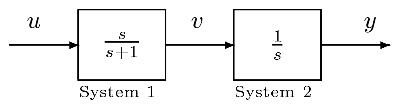

Figure 2: Cascade System: Reachability Issues and Control Construction

Problem 11 (Making a Mode Uncontrollable by Introducing a Pole-Zero Cancellation)
Consider the cascade system in Figure 3. The system P *has a state space representation*

$$\dot{x}_p = A_p x_p + B_p u \tag{61}$$
$$y = C_p x_p + D_p u. \tag{62}$$

The system K *has a state space representation*

$$\dot{x}_k = A_k x_k + B_k e \tag{63}$$
$$u = C_k x_k + D_k e. \tag{64}$$

(a) Find a state space representation (A, B, C, D) *for the cascade system such that*

$$\dot{x} = Ax + Be \tag{65}$$
$$y = Cx + De \tag{66}$$

where $x = [\ x_k^T \quad x_p^T\ ]^T$.

(b) Suppose that $w^T(sI - A_p) = 0^T$ *with* $w \neq 0$*; i.e.* s *is a pole of* P *and* w *is a corresponding left eigenvector of* A_p*. Suppose there exists* $y_1 \neq 0$ *such that* $[\ y_1^T \quad -w^T B_p\ ]^T \neq 0^T$ *and*

$$\left[y_1^T \quad -w^T B_p \right] \begin{bmatrix} sI - A_k & -B_k \\ C_k & D_k \end{bmatrix} = 0^T. \tag{67}$$

Show that this implies that the mode s *in the plant* P *is uncontrollable from* e*.*

(c) Suppose that

$$A_p = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} \quad B_p = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 0 \end{bmatrix} \quad C_p = \begin{bmatrix} 4 & 0 & 5 \\ 0 & 6 & 0 \end{bmatrix} \tag{68}$$

$$A_k = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad B_k = \begin{bmatrix} 7 & 0 \\ 0 & 8 \\ 9 & 0 \end{bmatrix} \tag{69}$$

(c) Show that the plant P has a pole at $s = -1$.
(d) Choose C_k such that the plant pole at $s = -1$ is uncontrollable from e. Answer: Show that the general solution is

$$C_p = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \tag{70}$$

where

$$\begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -7/3 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -4/3 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} d + \begin{bmatrix} 0 \\ -4/3 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} e + \begin{bmatrix} 0 \\ 0 \\ -4/3 \\ 0 \\ 0 \\ 1 \end{bmatrix} f \tag{71}$$

where d, e, f are arbitrary real numbers. ∎

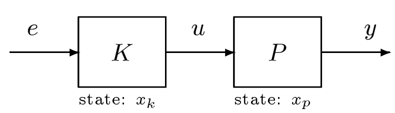

Figure 3: Cascade System: Making A Mode Uncontrollable Through a Pole-Zero Cancellation

Problem 12 (Extraction of a MIMO Transmission Zero)
Let P represent a multiple-input multiple-output (MIMO) system with a transmission zero at λ. Suppose that

$$\begin{bmatrix} \lambda I - A & -B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{72}$$

where $u^T u = 1$. Define

$$G(s) \stackrel{\text{def}}{=} [A, \hat{B}, C, D] \tag{73}$$

$$\hat{B} \stackrel{\text{def}}{=} B - 2\lambda x u^T \tag{74}$$

$$H(s) \stackrel{\text{def}}{=} [\,-\lambda,\ u^T,\ -2\lambda u,\ I\,] \tag{75}$$

(a) Show that H is all-pass ($\sigma_i[H(j\omega)] = 1$ for all ω and i). Hint: Show that $H^H(j\omega)H(j\omega) = I$ for all ω.
(b) Let e and y denote the input and output of the cascade system, respectively. Find a state space representation for the cascade system GH; i.e. the cascade system formed by H followed by G. Let x_1 be the state of H and x_2 be the state of G. Answer:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\lambda & 0 \\ -2\lambda \hat{B} u & A \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} u^T \\ \hat{B} \end{bmatrix} e \tag{76}$$

$$y = \begin{bmatrix} -2\lambda D u & C \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + De \tag{77}$$

(c) Use the result from (b) to comput the transfer function of the cascade system. Show that $P(s) = G(s)H(s)$.

■

Problem 13 (Full State Feedback, Stabilizability)
Consider the cascade system in Figure 4. Suppose that system 1 has a state space representation

$$\dot{x}_1 = u \tag{78}$$

$$v = -x_1 + u \tag{79}$$

and system 2 has a state space representation

$$\dot{x}_2 = x_2 + v \tag{80}$$

$$y = x_2 \tag{81}$$

(a) Find a state space representation (A, B, C, D) for the cascade system.
(b) Is the system controllable? Explain your answer.
(c) Consider the full state control law $u = -Gx$ where $G = [\, g_1 \;\; g_2 \,]$. What is the resulting closed loop characteristic polynomial? What are the resulting closed loop poles? Is the system (A, B, C, D) stabilizable via full state feedback? Explain.

■

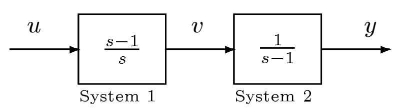

Figure 4: State Feedback and Stabilizability Issues

Problem 14 (Derivation of LQ Frequency Domain Equality)
Suppose that K satifies the following Control Algebraic Riccati Equation (CARE)*:*

$$0 = -KA - A^T K - M^T M + KB\frac{1}{\rho}B^T K \tag{82}$$

Suppose we define

$$G_{LQ}(s) \stackrel{\text{def}}{=} G\Phi(s)B \tag{83}$$

$$G_{OL}(s) \stackrel{\text{def}}{=} M\Phi(s)B \tag{84}$$

where $G \stackrel{\text{def}}{=} \frac{1}{\rho}B^TK$ *and* $\Phi(s) \stackrel{\text{def}}{=} (sI - A)^{-1}$. *We want to prove the so-called LQ Frequency Domain Equality (LQFDE):*

$$[\, I + G_{LQ}(-s)\,]^T \ [\, I + G_{LQ}(s)\,] = I + \frac{1}{\rho}G_{OL}(-s)^T G_{OL}(s). \tag{85}$$

Hints (Algebraic):
(i) Add and subtract KsI *to show that* $0 = K(sI - A) + (-sI - A^T)K - M^TM + KB\frac{1}{\rho}B^TK$.
(ii) Pre-multiply by $\frac{1}{\sqrt{\rho}}B_T(-sI - A^T)^{-1}$ *and post-multiply by* $(sI - A)^{-1}B\frac{1}{\sqrt{\rho}}$.
(iii) Substitute in for G, G_{LQ}, *and* G_{OL}.
(iv) Show that $0 = G_{LQ}(-s)^T + G_{LQ}(s) - G_{OL}(-s)^T\frac{1}{\rho}G_{OL}(s) + G_{LQ}(-s)^TG_{LQ}(s)$.
(v) Add the identity matrix to both sides. The result then follows.

∎

Problem 15 (LQR Problem For Double Integrator)
Consider the double integrator system in Figure 5.
(a) Determine the state space representation (A, B, M) *for the system.*
Answer: $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $M = [\, 1 \quad 0\,]$.
(b) Determine the control law which minimizes the following quadratic cost functional:

$$J(u) = \int_0^\infty (z^2 + \rho u^2)\, dt \tag{86}$$

Partial Answer:
$u = -Gx$ *where* $G = \frac{1}{\rho}B^TK = [\, g_1 \quad g_2\,]$ *and* K *is the unique symmetric positive definite solution of the Control Aglebraic Riccati Equation (CARE):*

$$0 = -KA - A^TK - M^TM + KB\frac{1}{\rho}B^TK. \tag{87}$$

Let $K = \begin{bmatrix} k_1 & k_2 \\ k_2 & k_3 \end{bmatrix}$. *Show that*

$$0 + 0 + 1 = \frac{1}{\rho}k_2^2 \tag{88}$$

$$k_1 + 0 + 0 = \frac{1}{\rho}k_2k_3 \tag{89}$$

$$k_2 + k_2 + 0 = \frac{1}{\rho}k_3^2 \tag{90}$$

Find k_2, k_3 *and then* k_1. *Show that*

$$k_1 = \sqrt{2}\rho^{1/4} \tag{91}$$

$$k_2 = \sqrt{\rho} \tag{92}$$

$$k_3 = \sqrt{2}\rho^{3/4} \tag{93}$$

Determine g_1 *and* g_2. *Show that*

$$g_1 = \frac{1}{\sqrt{\rho}} \tag{94}$$

$$g_2 = \frac{\sqrt{2}}{\rho^{1/4}} \tag{95}$$

(c) Determine the closed loop characteristic equation. Determine the closed loop poles.
Answer: $\Phi_{CL}(s) = det(sI - A + BG) = s^2 + \frac{\sqrt{2}}{\rho^{1/4}} s + \frac{1}{\sqrt{\rho}}$. *Poles:* $s = \left[-\frac{1}{\sqrt{2}} \pm j\frac{1}{\sqrt{2}}\right]\frac{1}{\rho^{1/4}}$.

(d) Compute G_{LQ}. *Answer:* $G_{LQ} = \frac{\sqrt{2}}{\rho^{1/4}}\left[\frac{s+\frac{1}{\sqrt{2}\rho^{1/4}}}{s^2}\right]$.

(e) Show that the resulting feedback loop - when broken at the plant input - has an infinite upward gain margin and a downward gain margin of zero.

(f) Show that the approximate open loop gain crossover frequency ($|G_{LQ}| = 1$) *is* $\omega_g \approx \frac{\sqrt{2}}{\rho^{1/4}}$ *and that the approximate phase margin is* $PM \approx 63.43°$. *Hint: The gain crossover frequency occurs on the high frequency Bode asymptote of* G_{LQ} *where the slope is -20 dB/decade.*

(g) Now consider the rearranged LQ-servo loop in Figure 6. What is the transfer function L_e *from e to y? Is* $L_e = G_{LQ}$*? What are the upward and downward gain margins? Why? Base your justification on a root locus for* L_e. *What is the approximate gain crossover frequency? What is the approximate phase margin?*

Answers: $L_e = \frac{g_1}{s(s+g_2)} = \frac{\frac{1}{\sqrt{\rho}}}{s(s+\frac{\sqrt{2}}{\rho^{1/4}})}$. *No. In general,* $L_e \neq G_{LQ}$. $\uparrow GM = \infty$, $\downarrow GM = 0$, $\omega_{g_e} \approx \frac{1}{\sqrt{2}\rho^{1/4}}$, $PM = 63.43°$

(h) Provide numerical justification for the above formulae for K *and* G *by computing everything numerically for* $\rho = 1$ *and* $\rho = 0.01$ *using MATLAB and comparing the computer results with those determined analytically above.*

■

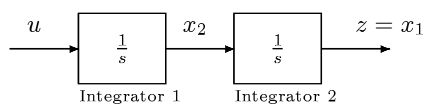

Figure 5: Double Integrator System

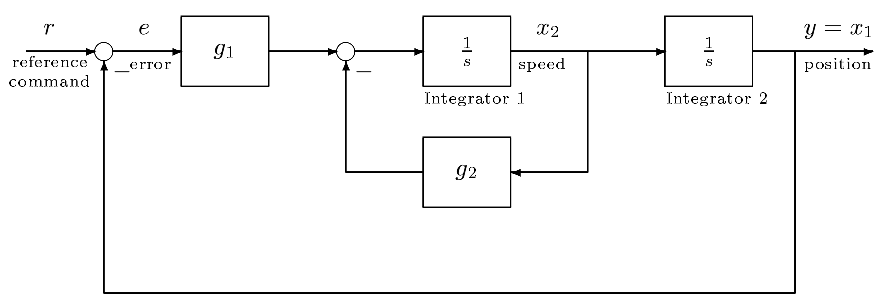

Figure 6: LQ-Servo Loop For Double Integrator

Problem 16 (LQR Servo Design for Modified F8 Aircraft)

Consider a modified F8 aircraft with the following longitudinal dynamics

$$\dot{x}_1 = A_1 x_1 + B_1 u_1 \quad (96)$$

$$y_1 = C_1 x_1 \tag{97}$$

where

$$A_1 = \begin{bmatrix} -0.8 & -0.0006 & -12 & 0 \\ 0 & -0.014 & -16.64 & -32.2 \\ 1 & -0.0001 & -1.5 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{98}$$

$$B_1 = \begin{bmatrix} -19 & -2.5 \\ -0.66 & -0.5 \\ -0.16 & -0.6 \\ 0 & 0 \end{bmatrix} \tag{99}$$

$$C_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \tag{100}$$

$$u_1 = \begin{bmatrix} \delta_e & \text{elevator deflection from trim (rad)} \\ \delta_f & \text{flaperon deflection (rad)} \end{bmatrix} \tag{101}$$

$$x_1 = \begin{bmatrix} q & \text{pitch rate (rad/sec)} \\ v & \text{perturbation from equilibrium horizontal speed } (ft/sec) \\ \alpha & \text{perturbed angle of attack from trim (rad)} \\ \theta & \text{perturbed pitch angle from trim (rad)} \end{bmatrix} \tag{102}$$

$$y_1 = \begin{bmatrix} \theta & \text{perturbed pitch angle from trim (rad)} \\ \gamma & \text{perturbed flight path angle from trim (rad)} \end{bmatrix} \tag{103}$$

Now lets perform a coordinate transformation as follows:

$$u = S_u u_1 \tag{104}$$

$$x = S_x x_1 \tag{105}$$

$$y = S_y y_1 \tag{106}$$

where

$$S_u = \begin{bmatrix} r2d & 0 \\ 0 & r2d \end{bmatrix} \tag{107}$$

$$S_x = \begin{bmatrix} 0 & 0 & 0 & r2d \\ 0 & 0 & -r2d & r2d \\ r2d & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{108}$$

$$S_y = \begin{bmatrix} r2d & 0 \\ 0 & r2d \end{bmatrix} \tag{109}$$

and $r2d = 180/\pi$ *is a radian to degree conversion factor. The new variables (and units) are then as follows:*

$$u_p = \begin{bmatrix} \delta_e & \text{elevator deflection from trim (deg)} \\ \delta_f & \text{flaperon deflection (deg)} \end{bmatrix} \tag{110}$$

$$x_p = \begin{bmatrix} \theta & \text{perturbed pitch angle from trim (deg)} \\ \gamma & \text{perturbed flight path angle from trim (deg)} \\ q & \text{pitch rate (deg/sec)} \\ v & \text{perturbation from equilibrium horizontal speed } (ft/sec) \end{bmatrix} \tag{111}$$

$$y_p = \begin{bmatrix} \theta & \text{perturbed pitch angle from trim (deg)} \\ \gamma & \text{perturbed flight path angle from trim (deg)} \end{bmatrix} \tag{112}$$

(a) What is the new state space representation A_p, B_p, C_p?

We now want to develop a control system based on LQR theory.

(b) In order to guarantee zero steady state error to step reference commands, we begin by augmenting the plant with integrators at the plant output. Eventually these integrators will be moved around the loop and become part of the compensator to be designed. Determine the state equation

$$\dot{x} = Ax + Bu \tag{113}$$

where

$$x = \begin{bmatrix} z \\ x_p \end{bmatrix} = \begin{bmatrix} z \\ y \\ x_r \end{bmatrix} \tag{114}$$

z represents the integrator states and $x_r = [\ q \ \ v\ ]^T$ contains the rest of the state variables in x_p other than those in $y = [\ \theta \ \ \gamma\ ]^T$.

(c) Determine the optimal control law that minmizes the following quadratic cost functional:

$$J(u) = \int_0^\infty (x^T Q x + u^T R u)\, dt \tag{115}$$

where $Q = diag(\ 1,\ 1,\ 1,\ 1,\ 0,\ 0\ )$ and $R = 0.01 I_{2\times 2}$. Note that the state weighting matrix Q penalizes the two integrator states and the two outputs θ and γ, but not q and v. Answer: $u = -Gx$ where

$$G = [\ g_z \ \ g_y \ \ g_r\ ] \tag{116}$$

$$= \begin{bmatrix} -9.8028 & -1.9764 & -10.6912 & -2.4591 & -1.0367 & 0.0006 \\ -1.9764 & 9.8028 & -0.0103 & 9.2501 & 0.0191 & 0.0094 \end{bmatrix}. \tag{117}$$

$$\tag{118}$$

Form LQ Servo. *We now modify the structure of the LQ loop to accomodate reference commands and to ensure zero steady state error to step reference commands. To do this, we begin by noting that*

$$u = -g_z z - g_y y - g_r x_r \tag{119}$$

may be visualized as shown in Figure 7 with $r = 0$, where $C_r = [\ 0_{2\times 2} \ \ I_{2\times 2}\ ]$ selects x_r from $x_p = [\ y^T \ \ x_r^T\ ]^T$. Figure 7 shows where the reference command may be naturally injected. In this figure, $ed = r - y$ is the so-called tracking error.

(d) Determine a state space representation for the transfer function matrix from e to y; i.e. for the open loop transfer function matrix obtained by breaking the loop at the error signal e.
(e) Determine a state space representation for the transfer function matrix from r to y.
(f) Evaluate the ability of the closed loop system to follow step reference commands. Consider the following reference command scenario: $r = [\,1 \;\; 0\,]^T$, $r = [\,0 \;\; 1\,]^T$, $r = [\,1 \;\; 1\,]^T$, $r = [\,-1 \;\; 1\,]^T$. For each case plot the outputs y and the controls u.
(g) Plot the open loop singular values when the loop is broken at the plant input and when the loop is broken at the plant output or error signal. How and why do the two plots differ?
(h) Repeat (g) for the associated sensitivity singular values.
(i) Repeat (g) for the associated complementary sensitivity singular values.
(j) Use the closed loop sensitivity and complementary sensitivity singular values (associated with the plant output or error signal) to explain the behavior seen in the plots in (f). Explain how bumps in the complementary sensitivity affect the overshoots seen in the outputs y.
(k) What transfer function matrix would a multiplicative modeling error reflected at the plant input see? Find a state space representation for it. Plot the singular values of this transfer function matrix.
(l) What transfer function matrix would a multiplicative modeling error reflected at the plant output see? Find a state space representation for it. Plot the singular values of this transfer function matrix.

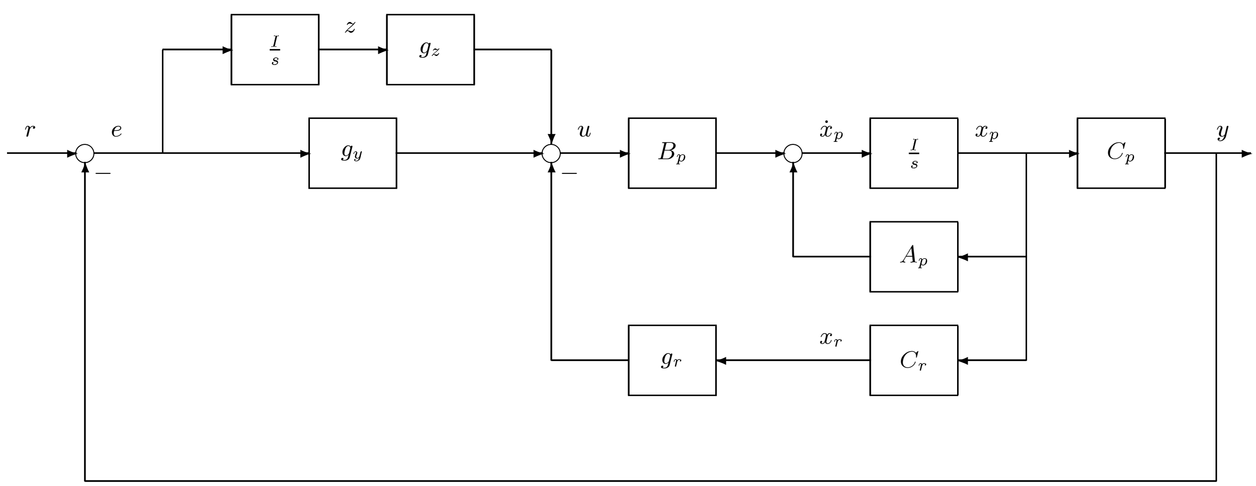

Figure 7: LQ Servo: Adapting LQR Control Law For Reference Command Following

Problem 17 (LQR Problem for Generic Helicopter)
Consider the following dynamical model for a generic single rotor helicopter:

$$\dot{x} = Ax + Bu \tag{120}$$
$$y = Cx \tag{121}$$

where

$$A = \begin{bmatrix} -0 & 1 & 0 & 0 \\ 0 & -0.414 & -0.011 & 0 \\ 9.80 & -1.43 & -0.02 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad (122)$$

$$B = \begin{bmatrix} 0 \\ 6.27 \\ 9.8 \\ 0 \end{bmatrix} \quad (123)$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \quad (124)$$

$$u = \begin{bmatrix} \delta \ \text{ longitudinal cyclic pitch control (rad)} \end{bmatrix} \quad (125)$$

$$x = \begin{bmatrix} \theta \ \text{ pitch attitude (rad)} \\ q \ \text{ pitch rate (rad/sec)} \\ \dot{z} \ \text{ horizontal speed (m/sec)} \\ z \ \text{ horizontal position (m)} \end{bmatrix} \quad (126)$$

$$y = \begin{bmatrix} z \ \text{ horizontal position (m)} \end{bmatrix} \quad (127)$$

(a) What are the system poles and zeros?
(b) Use MATLAB to plot the system frequency response.
(c) Determine the optimal control law which minimizes the following quadratic cost functional:

$$J(u) = \int_0^\infty (y^2 + \rho u^2)\, dt \quad (128)$$

where $\rho = 400$ *is the control weighting parameter. Answer:* $u = -Gx$, $G = [\ 0.9088 \ \ 0.3276 \ \ 0.0944 \ \ 0.0500\ ]$.
(d) Use MATLAB to plot the frequency response for $G_{LQ} = G(sI - A)^{-1}B$. *What is open loop transfer function* G_{LQ} *- i.e. open loop transfer function when the loop is broken at the plant input? What are the upward and downward gain margins* $\uparrow GM$, $\downarrow GM$ *? What are the phase and gain crossover frequencies* ω_p, ω_g*? What is the phase margin* PM*? What is the delay margin? Answers:* $\uparrow GM = \infty$, $\downarrow GM = 0.4022$, $\omega_p = 1.1993$ *rad/sec*, $\omega_g = 2.9327$ *rad/sec*, $PM = 60.0108°$, $DM = 0.3571$ *sec.*
(e) What are the closed loop poles? What are the associated damping factors and undamped natural frequencies?

LQ Servo and Command Following. *Now suppose that the LQ design is "rearranged" to accomodate reference commands. To do this, we wish to replace the following closed loop dynamical system*

$$\dot{x} = Ax + Bu = Ax - BGx = Ax - B[\ g_1 \ \ g_2 \ \ g_3 \ \ 0\ ]x - Bg_4 y \quad (129)$$

with the following

$$\dot{x} = Ax + Bu = Ax - BGx = Ax - B[\ g_1 \ \ g_2 \ \ g_3 \ \ 0\ ]x + Bg_4(r - y). \quad (130)$$

Here, r *represents a reference command that has been introduced in the "appropriate" place. The error signal* $e = r - y$ *is the so-called tracking error.*

(f) Use MATLAB to plot the frequency response for L_e - the open loop transfer function when the loop is broken at the error signal. This is the transfer function from $e = r - y$ to y. Determine L_e? What are the associated upward and downward gain margins $\uparrow GM$, $\downarrow GM$? What are the associated phase and gain crossover frequencies ω_p, ω_g? What is the associated phase margin PM? What is the associated delay margin? How do these compare with thos obtained in (d) - when the loop is broken at the plant input? Answers: $\uparrow GM = 2.4784$ (Why is it finite?), $\downarrow GM = 0$ (How can this be if the plant is unstable? Draw a block diagram for the closed loop system. Note the inner loop. What are the open loop poles of L_e?), $\omega_p = 1.2478$ rad/sec, $\omega_g = 0.5231$ rad/sec, $PM = 60.0227°$, $DM = 2.0028$ sec.
(g) Plot the sensitivity frequency response associated with the loop broken at the plant input; i.e. $S_{LQ} = \frac{1}{1+G_{LQ}}$. How large does it get in magnitude? What does this imply in terms of stability margins at the plant input?
(h) Plot the sensitivity frequency response associated with the loop broken at the plant output or error signal; i.e. $S_e = \frac{1}{1+L_e}$. How large does it get in magnitude? What does this imply in terms of stability margins at the plant output or error signal?
(i) Plot the frequency response for the closed loop transfer function (from r to y). What does it tell you about low frequency command following? What is the approximate closed loop bandwidth? What are the closed loop poles and zeros? What are the damping factors associated with the closed loop poles? Are they good? Why? Partial Answer: $BW \approx 1$ rad/sec.
(j) Use MATLAB to compute the step response of the closed loop system. Plot the horizontal position y in meters, the pitch attitude θ in degrees, and the control u in degrees. What is the percent overshoot?
(k) How can the overshoot in your design be improved? Hint: Try a cost function that penalizes the horizontal position less; e.g.

$$J(u) = \int_0^\infty (z^2 + \rho u^2)\, dt \tag{131}$$

with $z = Mx$, $M = [\ 0\ \ 0\ \ 0\ \ 0.012\ ]$, and $\rho = 400$. What tradeoffs do you observe? What is the new overshoot? Settling time? Closed loop bandwidth?

■

Problem 18 (LQR Property For Minimum Phase Design Plants)
It can be shown that if $G_{OL} = M(sI - A)^{-1}B$ is minimum phase (i.e. has no right half plane transmission zeros), then

$$\lim_{\rho \to 0^+} K_\rho = 0 \tag{132}$$

where K_ρ is the unique symmetric positive semi-definite solution of the Control Algebraic Riccati Equation (CARE):

$$0 = -K_\rho A - A^T K_\rho - M^T M + K_\rho B \frac{1}{\rho} B^T K_\rho. \tag{133}$$

The subscript ρ on K_ρ emphasizes that K_ρ is a function of the control weighting (bandwidth) parameter ρ.

Suppose that $G_{OL} = M(sI - A)^{-1}B$ *is minimum phase (i.e. has no right half plane transmission zeros) and that* $G = \frac{1}{\rho}B^T K_\rho$*. Show that*

$$\lim_{\rho \to 0^+} \sqrt{\rho} G = WM \tag{134}$$

for some orthonormal matrix W *(i.e.* $W^T W = I$*).*
Hint: Focus on the CARE and use the following result from linear algebra:

$U^T U = W^T W$ *if and only if* $U = VW$ *for some orthonormal matrix* V*.*

■

Try to state the dual of the above results; i.e. the Filter Algebraic Riccati Equation (FARE) version.

Problem 19 (Model Based Compensator Design: Dynamic Augmentation and Pole Placement)
Consider the following unstable nonminimum phase plant

$$P = 2\left[\frac{s-5}{s-10}\right] = [A_p, B_p, C_p, D_p] \tag{135}$$

where $A_p = 10$*,* $B_p = 10$*,* $C_p = 1$*,* $D_p = 2$*.*

Design a 3rd order model based compensator K *which satisfied the following closed loop specifications:*

- *closed loop system exhibits zero steady state error to step commands and step disturbances;*
- *closed loop system has closed loop poles at* $s = -5 \pm j5$ *and* $s = -75 \pm j75$*.*

Hint (Design Process):
(i) Augment the plant with integrators to obtain the design plant $P_d = [A, B, C]$*:*

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 10 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = [\, -10 \quad 2 \,]. \tag{136}$$

The design plant P_d *is a fictitous "plant" that is used in the design process only - it is not physically built.*
(ii) Choose $G = [\, g_1 \;\; g_2 \,]$ *and* $H = [\, h_1 \;\; h_2 \,]^T$ *in the model based compensator* $K_d(s) = [A - BG - HC, H, G]$ *so that the closed loop system with* K_d *and* P_d *satisfies the pole placement specification. Recall from the so-called* separation principle *that the closed loop poles for the resulting feedback system are precisely the eigenvalues of* $A - BG$ *(due to state feedback) and those of* $A - HC$ *(due to the observer or state estimator).*
(iii) The final model based compensator design is then given by $K(s) = \frac{K_d(s)}{s}$*. It is the compensator* K *that is physically built to work with the actual plant* P *in a negative unity feedback loop.*

■

Problem 20 (Dynamic Augmentation and Shaping)
Consider the dyanmical system shown in Figure 8. The figure depicts a nominal plant model with state space representation given by

$$\dot{x}_p = A_p x_p + B_p u_p \quad (137)$$
$$y_p = C_p x_p. \quad (138)$$

The plant has been augmented with integrators in each control channel at the plant input u_p. *The matrices* L_L *and* L_H *should be thought of as design parameters used to do loop shaping (e.g. for designing a Kalman filter loop with required multivariable directionality properties).*

(a) Define the augmented state vector as follows: $x = \begin{bmatrix} x_p \\ x_i \end{bmatrix}$. *Determine a state space representation for the augmented system; i.e. find* (A, B, L) *such that*

$$\dot{x} = Ax + L\xi \quad (139)$$
$$y = Cx. \quad (140)$$

Note: The matrix L *will be used as a design parameter in the design of Kalman filter loops with prescribed multivariable directionality properties.*
(b) Determine the transfer function matrix $G_{FOL} = C(sI - A)^{-1}L$ *from* ζ *to* y. *Hint: Follow the flow in Figure 8 or use the following result from linear algebra:*

$$\begin{bmatrix} M & N \\ 0 & P \end{bmatrix}^{-1} = \begin{bmatrix} M^{-1} & -M^{-1}NP^{-1} \\ 0 & P^{-1} \end{bmatrix}. \quad (141)$$

(142)

(c) Approximate G_{FOL} *at low frequencies (small* s*).*
(d) Approximate G_{FOL} *at high frequencies (large* s*).*

(e) Matching at Low Frequencies. *Assume that the nominal plant model* (A_p, B_p, C_p) *has no natural integrators in it; i.e.* A_p *has no zero eigenvalues and hence is invertible. Also assume that its dc gain matrix is nonsingular (i.e. invertible). Determine* L_L *such that* $G_{FOL} \approx \frac{I}{s}$ *for small* s. *Such an* L_L *matches the singular values of* G_{FOL} *at low frequencies.*
Answer: $L_L = [C_p(-A_p)^{-1}B_p]^{-1} U_L$ *where* U_L *is an orthonormal matrix (i.e.* $U_L^T U_L = I$*).*

(f) Matching at High Frequencies. *Assume that* C_p *has full row rank. This guarantees the existance of a right inverse for* C_p. *Determine* L_H *such that* $G_{FOL} \approx \frac{I}{s}$ *for large* s. *Such an* L_H *matches the singular values of* G_{FOL} *at high frequencies.*
Answer: $L_H = C_p^T \begin{bmatrix} C_p & C_p^T \end{bmatrix}^{-1} U_H$ *where* U_H *is an orthonormal matrix (i.e.* $U_H^T U_H = I$*).*

(g) Matching at All Frequencies. *Show how to choose* L_L *and* L_H *such that* $G_{FOL} = \frac{I}{s}$ *for all* s. *Such a selection would match the singular values of* G_{FOL} *at all frequencies.*
Answer: $L_L = [C_p(-A_p)^{-1}B_p]^{-1} U_L$, $L_H = -A_p^{-1}B_p L_L$, *where* U_L *is an orthonormal matrix (i.e.* $U_L^T U_L = I$*).*
■

Comment: (Kalman Filter Loop Shaping)
The above loop shaping ideas for

$$G_{FOL}(s) = C(sI - A)^{-1}L \tag{143}$$

are fundamental for shaping Kalman filter loops:

$$G_{KF}(s) = C(sI - A)^{-1}H \tag{144}$$

where

$$H = \Sigma C^T \frac{1}{\mu} \tag{145}$$

and Σ is the unique symmetric positive semi-definite ($\Sigma = \Sigma^T \geq 0$) solution of the following *Filter Algebraic Riccati Equation (FARE)*:

$$0 = A\Sigma + \Sigma A^T + LL^T - \Sigma C^T \frac{1}{\mu} C\Sigma. \tag{146}$$

To guide us in the selection of the loop shaping matrix L and the bandwidth parameter μ, we have the *Kalman Fliter Frequency Domain Equality (KF-FDE):*

$$[\ I + G_{KF}(s)\ ]\ [\ I + G_{KF}(-s)\ ]^T = I + \frac{1}{\mu} G_{FOL}(s) G_{FOL}(-s)^T \tag{147}$$

From this, it follows that

$$\sigma_i\left[I + G_{KF}(j\omega)\right] = \sqrt{1 + \frac{1}{\mu}\sigma_i^2\left[G_{FOL}(j\omega)\right]}. \tag{148}$$

∎

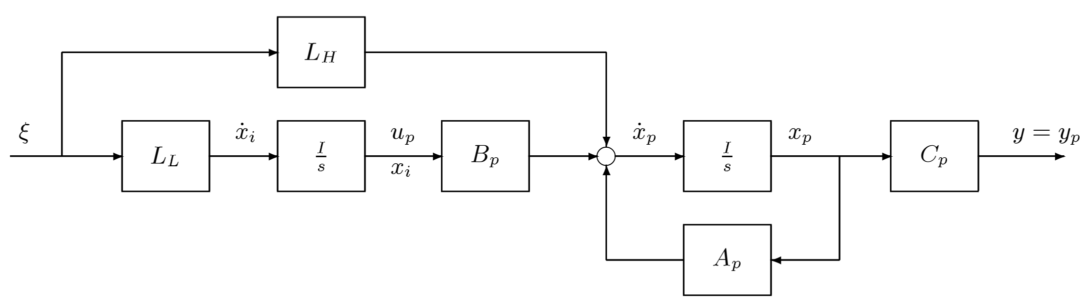

Figure 8: System Used To Explore Dynamic Augmentation and Shaping

Problem 21 (Loop Transfer Recovery (LTR) at the Plant Output)
Consider the standard negative feedback system in Figure 9 with design plant $P_d(s) = [A, B, C]$ *and compensator* $K_d(s, \rho) = [A - BG_\rho - HC, H, G_\rho]$ *in the feedback loop. The compnesator* K_d *may be visualized as shown in Figure 10, where* $\Phi(s) \stackrel{\text{def}}{=} (sI - A)^{-1}$. *Let* $x = [\ x_p^T\ \ x_k^T\ ]$ *denote the state of the cascade system formed by* K_d *followed by* P.

(a) Determine the the state space respresentation for the open loop system formed by breaking the loop at the plant output y or error signal e. Hint: $L_e = P_d K_d$.

(b) Determine the the state space respresentation for the open loop system formed by breaking the loop at the (so-called innovations or residual) signal v. Hint: $L_v = G_{KF} \stackrel{\text{def}}{=} C(sI - A)^{-1}H$.

(c) Suppose that

$$\lim_{\rho \to 0^+} \sqrt{\rho} G_\rho = WC \tag{149}$$

for some orthonormal matric W (i.e. $W^T W = I$). Show that this limiting result involving G_ρ and C implies the following loop transfer recovery result at the plant output or error signal.

$$\lim_{\rho \to 0^+} P_d(s) K_d(s, \rho) = G_{KF} \stackrel{\text{def}}{=} C(sI - A)^{-1}H. \tag{150}$$

Hint: We want to prove this important result using simple block diagram manipultations. The alternative is to apply the "ugly" Matrix Inversion Lemma

$$(A + BCD)^{-1} = A^{-1} - A^{-1}B\left[C^{-1} + DA^{-1}B\right]^{-1} DA^{-1} \tag{151}$$

twice!
To prove the result, substitute $G_\rho \approx \frac{1}{\sqrt{\rho}} WC$ into the model based compensator block diagram in Figure 10. Move the term $\frac{1}{\sqrt{\rho}} C$ leftward next to $\Phi(s) = (sI - A)^{-1}$. To maintain block diagram signal correctness, we must replace C in the feedback path by $\sqrt{\rho} I$. Now let $\rho \to 0$. This destroys the feedback within the compensator. Compute the resulting transfer function matrix from w to u. The result then follows.
Note: Letting ρ approach zero effectively destroys the feedback in the model based compensator and transfers the loop properties at v to e or equivalently to y.

■

Note: A similar result holds for loop transfer recovery at the plant input. Can you state it?

Comment: (Sufficient Condition for Limiting LTR Result)
It can be shown that a sufficient condition for Equation 149 is that the plant $[A, B, C]$ is minimum phase; i.e. has no right half plane zeros. This condition, while sufficient, is not necessary. ■

Problem 22 (LTR at Plant Input for a Non-Minimum Phase SISO Plant)
This problem examines loop transfer recovery at the plant input for the following non-minimum phase design plant

$$P_d(s) = \frac{1 - s}{s(s + 10)} = [A, B, C] \tag{152}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -10 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & -1 \end{bmatrix}. \tag{153}$$

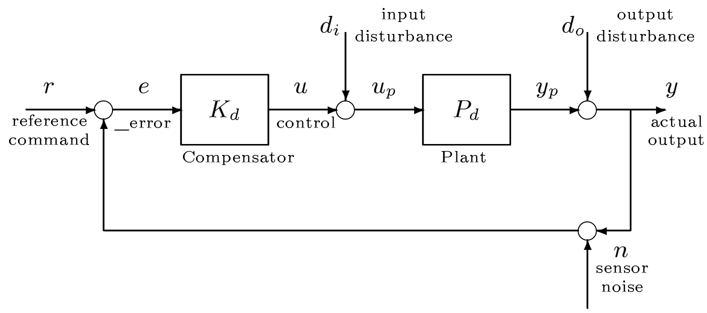

Figure 9: Standard Negative Feedback System

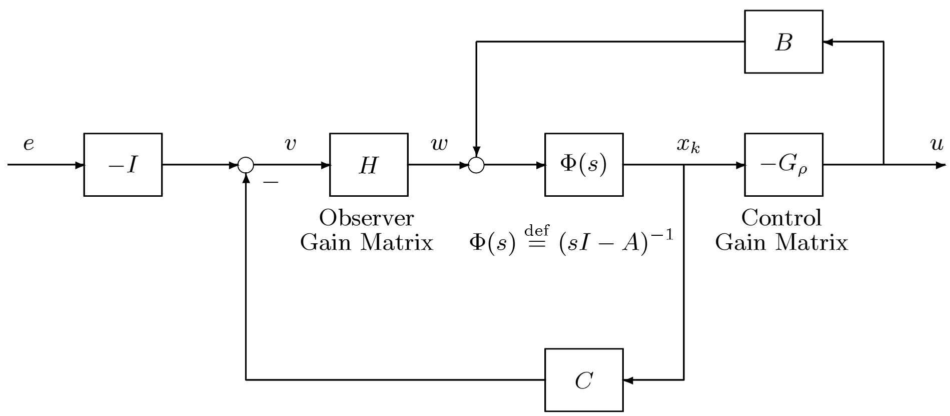

Figure 10: Block Diagram Visualization of Model Based Compensator

Consider a target loop (at the plant input) given by

$$L_i(s) = \frac{5(1-s)}{s(s+10)} \stackrel{\text{def}}{=} [A, B, G] \tag{154}$$

where $G = 5C$. Suppose we desire to recover this loop (at the plant input) using a model based compensator $K_d(s, \mu) = [A - BG - H_\mu C, H_\mu, G]$ with

$$H_\mu \quad = \quad \begin{bmatrix} 1 + \frac{1}{\mu^{1/4}} \\ -\frac{1}{\mu^{1/2}} \end{bmatrix} \tag{155}$$

(a) Show that

$$\lim_{\mu \to 0^+} \sqrt{\mu} H_\mu = BW \tag{156}$$

for some orthonormal matrix W (i.e. $W^T W = I$).
(b) Show that

$$\lim_{\mu \to 0^+} K_d(s, \mu) P_d(s) = L_i(s) \stackrel{\text{def}}{=} [A, B, G]. \tag{157}$$

■

This problem shows that we can achieve loop transfer recovery (LTR) when the right half plane "stuff" in the design plant P_d is appropriately incorporated into the target loop. This makes sense since we can't invert stuff in the right half plane! A necessary condition, therefore, for LTR is that the right half plane "stuff" associated with the design plant must be in the target loop.

Problem 23 (Loop Transfer Recovery (LTR) Design for CH-47 Helicopter) *Consider the following simplified CH-47 (tandem rotor) Helicopter model:*

$$\dot{x} = Ax + Bu \tag{158}$$

$$y = Cx \tag{159}$$

where

$$A = \begin{bmatrix} -0.02 & 0.005 & 2.4 & -32 \\ -0.14 & 0.44 & -1.3 & -30 \\ 0 & 0.018 & -1.6 & 1.2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{160}$$

$$B = \begin{bmatrix} 0.14 & -0.12 \\ 0.36 & -8.6 \\ 0.35 & 0.009 \\ 0 & 0 \end{bmatrix} \tag{161}$$

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 57.3 \end{bmatrix} \tag{162}$$

For additional information on the CH-47, visit http://www.boeing.com/rotorcraft/military/ch47d/.

(a) Plot the plant singular values? What are the plant poles and transmission zeros? Relate them to your singular value plot.

Concept of a Target Loop. *Let the matrix*

$$H = \begin{bmatrix} 0.015807 & -0.24054 \\ 9.066 & -0.1761 \\ 0.0091371 & 0.22873 \\ -0.0030733 & 0.089338 \end{bmatrix} \tag{163}$$

define a target feedback system defined by the following closed loop dynamical system:

$$\dot{x} = Ax + He \tag{164}$$

$$e = r - y \tag{165}$$

$$y = Cx \tag{166}$$

where r represents a reference command and e a tracking error. This feedback loop defines a target open loop transfer function matrix

$$L_{target} \stackrel{\text{def}}{=} C(sI - A)^{-1}H \tag{167}$$

with implicit properties at the error signal e or equivalently at the plant output y. Lets analyze the associated target open loop and closed loop systems.

(b) Plot the target open loop L_{target}, sensitivity S_{target}, and complementary sensitivty T_{target} singular values. What are the target open loop poles and transmission zeros? What are the target closed loop poles? What are the zeros of S_{target} and T_{target}?

Recovery of Target Loop Using Model Based Compensator and Cheap Control. *Next, we want to design a model based compensator $y = K(s)e$ which, when inserted into a classical unity negative feedback loop with the plant $P(s) = C(sI - A)^{-1}B$, recovers the target loop $L_{target} = C(sI - A)^{-1}H$ to the extant possible. To do this, we will use a model based compensator K possessing the following dynamic structure:*

$$K(s, \rho) = G_\rho(sI - A + BG_\rho + HC)^{-1}H \tag{168}$$

where G_ρ is found by solving a cheap coontrol *LQR problem ($R = \rho I, \rho > 0$) whose soltuion is given by*

$$G_\rho = \frac{1}{\rho}B^T K_\rho \tag{169}$$

where K_ρ is the unique positive semi-definite solution of the following Control Algebraic Riccati Equation (CARE)*:*

$$0 = -K_\rho A - A^T K_\rho - C^T C + K_\rho B \frac{1}{\rho} B^T K_\rho \tag{170}$$

with $M = C$.

Next we recall that if $G_{OL} = M(sI - A)^{-1}B$ is minimum phase (i.e. has no right half plane zeros), then

$$\lim_{\rho \to 0^+} \sqrt{\rho} G_\rho = WC \tag{171}$$

for some orthononrmal matrix W (i.e. $W^T W = I$). Given this, we now state following recovery result.

Loop Transfer Recovery Result. *When there exists an orthonormal matrix W such that G_ρ satisfies an equation such as Equation 171, then*

$$\lim_{\rho \to 0^+} P(s)K(s, \rho) = L_{target} \stackrel{\text{def}}{=} C(sI - A)^{-1}H \tag{172}$$

A similar recovery result holds for the target sensitivity and complementary sensitivity transfer functions

$$S_{target} \stackrel{\text{def}}{=} [I + L_{target}]^{-1} \tag{173}$$

$$T_{target} \stackrel{\text{def}}{=} I - S_{target} = L_{target}[I + L_{target}]^{-1}. \tag{174}$$

(c) Show how the target singular values may be recovered one decade above crossover. Plot the resulting open loop singular values for PK and the target open loop singular values for L_{target}. What is the resulting $\rho > 0$? G_ρ?

(d) Plot the singular values for the resulting compensator $K(s, \rho)$. What are the resulting compensator poles and transmission zeros? How do these relate to your singular value plot for $K(s, \rho)$?

(e) What are the resulting open loop poles and transmission zeros? How do these compare with the target open loop poles and transmission zeros?
(f) Show that the transmission zeros of a model based compensator $K(s) = G(sI-A+BG+HC)^{-1}H$ *are identical to those of the state space triple* (A, H, G)*. Hint: Show that*

$$\begin{bmatrix} A - BG - HC & -H \\ G & 0 \end{bmatrix} = \begin{bmatrix} I & -B \\ 0 & I \end{bmatrix} \begin{bmatrix} A & -H \\ G & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ C & I \end{bmatrix} \tag{175}$$

(g) What are the resulting closed loop poles? Which closed loop poles are due to the target loop? Which closed loop poles are due to the regulator?
Hint: By the so-called separation principle*, the closed loop poles resulting from our model based compensation scheme are the eigenvalues of* $A - HC$ *and those of* $A - BG_\rho$.
(g) Plot the singular values of the resulting sensitivity function $S = [I + PK]^{-1}$ *and the target sensitivity function* $S_{target} \stackrel{\text{def}}{=} [I + L_{target}]^{-1}$.
(h) Plot the singular values of the resulting complementary sensitivity function $T = I - S$ *and the target complementary sensitivity function* $T_{target} \stackrel{\text{def}}{=} I - S_{target}$.

∎

Problem 24 (LQG/LTR Design for AV-8A Harrier Aircraft (Nonminimum Phase)) *Consider the following simplified model for the longitudinal dynamics of a AV-8A Harrier Aircraft at a medium speed flight condition:*

$$\dot{x}_p = A_p x_p + B_p u_p \tag{176}$$

$$y_p = C_p x_p \tag{177}$$

where

$$A_p = \begin{bmatrix} 0 & 1.0000 & 0 & 0 & 0 & 0 \\ -1.8370 & -1.8930 & 1.8370 & -0.0004 & 0.0062 & -0.1243 \\ 0.5295 & 0.0085 & -0.5295 & 0.0006 & 0.0002 & 0.0017 \\ -34.5000 & 0 & 2.3000 & -0.0621 & 0.4209 & -0.0452 \\ 0 & 0 & 0 & 0 & -1.9660 & 0 \\ 0 & 0 & 0 & 0 & 0 & -12.0000 \end{bmatrix} \tag{178}$$

$$B_p = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1.9660 & 0 \\ 0 & 12.0000 \end{bmatrix} \tag{179}$$

$$C_p = \begin{bmatrix} 0 & 0 & 0 & 1.0000 & 0 & 0 \\ 0 & 0 & 57.2958 & 0 & 0 & 0 \end{bmatrix} \tag{180}$$

$$u_p = \begin{bmatrix} \textit{stick input} \\ \textit{throttle} \end{bmatrix} \tag{181}$$

$$x_p = \begin{bmatrix} \theta \ - pitch\ angle \\ q \ - pitch\ rate \\ \gamma \ - flight\ path\ angle \\ v \ - velocity \\ s_a \ - stabilizer\ angle \\ N \ - enginefan\ speed \end{bmatrix} \quad (182)$$

$$y_p = \begin{bmatrix} v \ - velocity \\ \gamma \ - flight\ path\ angle \end{bmatrix}. \quad (183)$$

(a) Design an LQG/LTR compensator K (loop transfer recovery at plant output) and pre-filter W such that the following specifications are satisfied by the closed loop system:

- *closed loop system is stable;*
- *closed loop system exhibits zero steady stae error to step reference commands and step disturbances;*
- *open loop singular values satisfy*

$$\sigma_i \left[P(j\omega)K(j\omega) \right] \geq 20\ dB \quad (184)$$

  for all frequencies $\omega \leq 0.085\ rad/sec$;

- *maximum singular value of the transfer function matrix T_{ny} from sensor noise n to output y satisfies*

$$\sigma_{max} \left[T_{ny}(j3) \right] \approx 1\ (0\ dB); \quad (185)$$

- *overshoot to a step velocity command does not exceed 10%;*
- *overshoot to a step γ command does not exceed 5%.*

Describe your design process and your final design parameters.
Hints:
(i) Augment plant with essential dynamics.
(ii) Design a target loop using Kalman filtering ideas. Choose your process noise matrix L so that the target open loop singular values are matched at low frequencies. Do not worry about high frequency loop shaping. Choose the sensor noise intensity μ to adjust the "bandwidth" of the target design and satisfy the low frequency PK singular value performance specification.
(iii) Solve a cheap LQR control problem to obtain $K_{mbc} = [\ A - BG - HC,\ H,\ G_\rho\ ]$ and attempt loop transfer recovery. Is loop transfer recovery possible? Why not? What are the plant transmission zeros? What are the transmission zeros of the target open loop transfer function matrix?
(iv) Form the final compensator K by combining K_{mbc} with the dynamic augmentation introduced earlier into the design plant.
(v) Design a pre-filter to satify the overshoot specifications.

(b) Plot the target and recovered open loop transfer function singular values.
(c) What are the target open loop pole and zeros? What are the recovered open loop poles and zeros? Observe and discuss near pole-zero cancellations that result between your compensator design K and the plant P.

(d) Plot the target and recovered closed loop transfer function singular values.
(e) Plot the target and recovered sensitivity transfer function singular values.
(f) Plot the singular values for the transfer function matrix from the reference command r to the controls u_p.
(g) Plot the outputs and controls for two command scenario: (1) velocity step command $r = [\,1 \;\; 0\,]^T$ and (2) flight path angle step command $r = [\,0 \;\; 1\,]^T$

■

Problem 25 (LQG/LTR Design for F404 Engine)
Consider the following simplified model for an F404 engine:

$$\dot{x} = Ax + Bu \tag{186}$$

$$y = Cx \tag{187}$$

where

$$A_p = \begin{bmatrix} -1.4600 & 3.3880 & 0 \\ 0.2219 & -2.2300 & 0 \\ 1.4670 & -4.8375 & -0.4000 \end{bmatrix} \tag{188}$$

$$B_p = \begin{bmatrix} 0.1840 & 0.4578 \\ 0.1630 & 0.0015 \\ 1.5325 & -0.0978 \end{bmatrix} \tag{189}$$

$$C_p = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{190}$$

(a) Design a model based LQG/LTR compensator such that the nominal closed loop system satisfies the following design specifications:

- *the nominal closed loop system is stable;*
- *the nominal closed loop system exhbits zero steady state error to step reference commands r and step output and input disturbances (do, di);*
- *the open loop singular values are matched at ALL frequencies and lie above 20 dB for all frequencies below 1 $rad/sec.$;*
- *the nominal closed loop system is robust with respect to a multiplicative uncertainty Delta at the plant output - the maximum singular value of Delta is bounded from above (strict inequality) by 0.1ω for all ω. NOTE: This will determine your desired open loop gain crossover frequency (bandwidth).*

(b) Describe your design process. What is L? What is the resulting H? mu? What is rho? Recover the open loop singular values 1 decade above crossover. What is G? Hints: Augment plant and CHOOSE L to MATCH AT ALL frequencies.
(c) What are the target open loop poles and transmission zeros?
(d) Plot the target open loop singular values (at plant output). Show that target singular values satisfy the robustness specification.
(e) What are target closed loop poles?
(f) Plot the target sensitivity and complementary sensitivity singular values.

(g) What are the poles and transmission zeros of the resulting compensator? Plot the compensator singular values?

(h) What are resulting closed loop poles? Which are due to the kalman filter? Which are due to the regulator? Hint: Separation principle.

(i) Plot the resulting sensitivity and complementary sensitivity singular values.Show that the resulting complementary sensitivity singular values satisfy the robustness specification. ∎