$x(t) = e^{At} x_0 \qquad x(0^-) = x_0$

$e^{At} = \sum_{i=1}^{n} e^{\lambda_i t} x_i y_i^H$

we assumed A has n L.I. eigenvectors x_i in order to get this.

$$x(t) = \sum_{i=1}^{n} e^{\lambda_i t} x_i \underbrace{(y_i^H x_0)}_{\text{scalar}}$$

$$= e^{\lambda_1 t} x_1 y_1^H x_0 + \cdots + e^{\lambda_k t} x_k y_k^H x_0 + \cdots + e^{\lambda_n t} x_n y_n^H x_0$$

Claim:

If $x_0 = x_k$, then $x(t) = e^{\lambda_k t} x_k$

"proof"

$$x(t) = e^{\lambda_1 t} x_1 y_1^H x_k + \cdots + e^{\lambda_k t} x_k y_k^H x_k + \cdots + e^{\lambda_n t} x_n y_n^H x_k$$

Recall:

$$X X^{-1} = I$$

$$\begin{bmatrix} — & y_1^H & — \\ & \vdots & \\ — & y_k^H & — \\ & \vdots & \\ — & y_n^H & — \end{bmatrix} \begin{bmatrix} | & & | \\ x_1 & \cdots & x_n \\ | & & | \end{bmatrix} = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$$

$$y_i^H x_i = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \qquad \begin{aligned} y_1^H x_1 &= 1 \\ y_1^H x_2 &= 0 \end{aligned}$$

$$x(t) = e^{\lambda_k t} x_k y_k^H x_k = e^{\lambda_k t} x_k$$ "end of Proof"

⇒ Example

$\dot{x} = Ax \qquad x(0^-) = x_0 \qquad A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

Let's study the natural modes of this system

a) <u>What are the eigenvalues</u> ?

$$\det(\lambda I - A) = 0$$

$$= \det \begin{bmatrix} \lambda & -1 \\ 2 & \lambda+3 \end{bmatrix} = \lambda(\lambda+3) - (-2)$$

$$= \lambda^2 + 3\lambda + 2$$

$$= (\lambda+1)(\lambda+2) = 0$$

$\boxed{\lambda_1 = -1}$ $\boxed{\lambda_2 = -2}$

* negative eigenvalues (poles) correspond to decaying exponentials in the time domain.

e^{-t} and e^{-2t}

These are <u>stable</u> modes

b) <u>Compute Right Eigenvectors</u>

→ will be linearly independent because two distinct eigenvalues.

$$(\lambda I - A)x = 0$$

$$\begin{bmatrix} \lambda_i & -1 \\ 2 & \lambda_i+3 \end{bmatrix} x = 0 \quad \rightsquigarrow \quad x = \begin{bmatrix} 1 \\ \lambda_i \end{bmatrix}$$

$$x_1 = \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad x_2 = \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Right Eigenvectors are <u>NOT</u> unique !

c) Compute Left Eigenvectors

Recall

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$X^{-1} = \begin{bmatrix} | & | \\ x_1 & x_2 \\ | & | \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}^{-1}$$

$$X^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} - & y_1^H & - \\ - & y_2^H & - \end{bmatrix}$$

d) Solve for $x(t)$

$$x(t) = e^{At}x_0 = \sum_{i=1}^{2} e^{\lambda_i t} x_i y_i^H x_0$$

$$= e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}[2 \;\; 1]\,x_0 + e^{-2t}\begin{bmatrix} 1 \\ -2 \end{bmatrix}[-1 \;\; -1]\,x_0$$

Note :

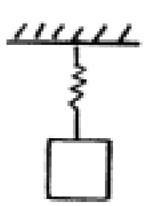
we could have found $x(t)$ by computing the inverse Laplace Transform of $(sI - A)^{-1}x_0 = X(s)$.

Analysis

If $x_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ then $x(t) = e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

If $x_0 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ then $x(t) = e^{-2t}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

Example



decaying exponential sinusoid