$R(A) \equiv$ column space of A

$$
\begin{aligned}
\dim R(A) &= \text{rank } A \\
&= \#\ \text{L.I. columns of } A \\
&= \#\ \text{L.I. Rows of } A \\
&= \#\ \text{of non zero pivots associated with } A \\
&= \#\ \text{of non zero singular values associated with } A \\
&= \text{dim of the largest non zero determinant} \\
&= r
\end{aligned}
$$

$$
\dim R(A^H) = r
$$

$$
\begin{aligned}
\dim \mathcal{N}(A) &= n - r \\
&= (\#\ \text{of variables}) - (\#\ \text{of basic variables}) \\
&= \#\ \text{of free variables}
\end{aligned}
$$

$$
\begin{aligned}
\dim \mathcal{N}(A^H) &= m - r \\
&= (\#\ \text{of equations}) - (\#\ \text{of basic variables}) \\
&= \#\ \text{of constraints on } b \text{ vector.}
\end{aligned}
$$

General Solution

Claim:

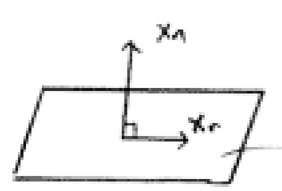
Any solution x to $Ax = b$ may be written as:

$$
x = x_r + x_n
$$

where $x_n \in \mathcal{N}(A)$, ie $Ax_n = 0$

$x_r \in R(A^H)$, ie $x_r = A^H v$ for some v

Visualize



$R(A^H)$ = row space

<u>Claim</u> $x_r \perp x_n$

<u>Check</u>
$$\begin{aligned} x_r^H x_n &= y^H A x_n \\ &= y^H 0 \\ &= 0 \end{aligned} \qquad x_r \perp x_n \;!$$

<u>Facts</u>

1) Elements in $\mathcal{N}(A)$ are $\perp$ (orthogonal) to elements in $R(A^H)$

<u>Idea:</u>

2) $R(A^H) = [\mathcal{N}(A)]^\perp$
3) $\mathcal{N}(A) = [R(A^H)]^\perp$

<u>Relationships</u>

$\mathcal{N}(A) = [R(A^H)]^\perp$ $\qquad$ $R(A^H) = [\mathcal{N}(A)]^\perp$

$R(A) = [\mathcal{N}(A^H)]^\perp$ $\qquad$ $\mathcal{N}(A^H) = [R(A)]^\perp$

- Two important $Ax = b$ problems

1) suppose $Ax = b$ has no solution
   Find x which minimizes the distance between b and Ax ~ $\|b - Ax\|$

   Least "Square" Error (LSE) Problem

2) Suppose $Ax = b$ has a solution but it is not unique.

Find a solution x which minimizes the size of x, $\|x\|$.

Minimum Norm Problem.

- <u>Solving LSE Problems</u>

$$\min_x \|b - Ax\|$$

inner product

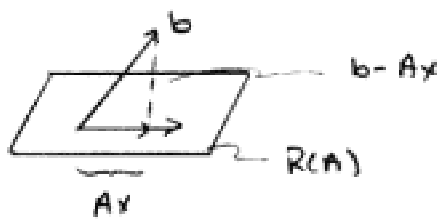
Define : $\langle y, z \rangle = z^H y$

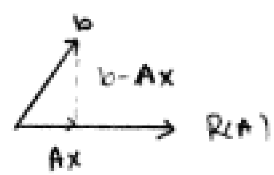
std euclidean norm $\rightarrow$ $\|z\| = \sqrt{\langle z, z \rangle} = \sqrt{z^H z}$

<u>solution</u>

x is a minimizing solution iff

$$b - Ax \perp R(A)$$





$b - Ax \perp R(A)$

$\perp Ay$ for all y

$y^H A^H (b - Ax) = 0$ for all y

$y^H (A^H b - A^H Ax) = 0$ for all y

$\therefore$ holds for $y = A^H b - A^H A x$

and

$y^H y = 0 \quad \therefore \quad y = 0 \qquad A^H A x = A^H b$