<u>Summary</u>

x is a minimizing solution to $\min \| b - Ax \|$ iff x satisfies the normal equations

$$A^H A x = A^H b$$

<u>Specific Case</u>

Suppose A has full column rank.

A has full column rank iff

- A has n L.I. columns
- A is one-to-one (injective)
- $\mathcal{N}(A) = \{0\}$
- $A^H A$ is invertible $\qquad A^H A \quad n \times n$

reason: $\mathcal{N}(A) = \mathcal{N}(A^H A)$

- A has a left inverse
- $A^{-L} A = I$, $\qquad A^{-L} = (A^H A)^{-1} A^H$

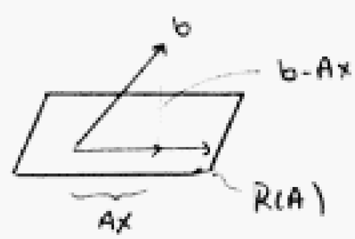
In such a case, $A^H A$ is invertible and the unique minimizing LSE solution is given by:

$$x = (A^H A)^{-1} A^H b$$

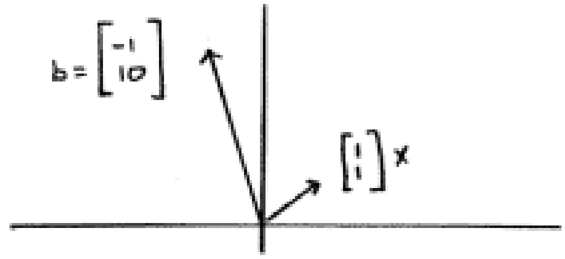
<u>Observation</u>

$$Ax = A (A^H A)^{-1} A^H b$$

$$= P_{R(A)} \cdot b$$



Example



$Ax = b$

$$A^H A x = A^H b$$

$$\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} x = \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 10 \end{bmatrix}$$

$$2x = -1 + 10 = 9$$
$$x = \frac{9}{2}$$

- Minimum Norm Problem

  Suppose $Ax = b$ has a solution but it is not unique

Given this, find an x which minimizes the norm of x, $\|x\|$

Solution

Fact: Any solution x may be written as

$$x = \underset{\text{row space}}{x_r} + \overset{\text{null space}}{x_n}$$

$$\|x\| = \sqrt{\|x_r\|^2 + \|x_n\|^2} \geq \|x_r\|$$

choose $x = x_r$!

Summary

x satisfies $Ax = b$ and minimizes $\|x\|$ iff

x satisfies

$$x = A^H y \text{ for some } y$$

$$\text{and } AA^H y = b$$

Note: $\mathcal{R}(A) = \mathcal{R}(AA^H)$

why?
$$\mathcal{N}(A^H) = \mathcal{N}(AA^H)$$
$$[\mathcal{N}(A^H)]^\perp = [\mathcal{N}(AA^H)]^\perp$$
$$\mathcal{R}(A) = \mathcal{R}(AA^H)$$

Specific Case

suppose A has full row rank

Facts:

A has full row rank iff A has m L.I. rows.

- $\mathcal{N}(A^H) = \{0\}$
- AA^H is invertible $\quad AA^H$ - $m \times m$
- A is onto (surjective)
- A has right inverse
- $AA^{-R} = I \quad A^{-R} = A^H(AA^H)^{-1}$

In such a case, AA^H is invertible and the unique y is given by $y = (AA^H)^{-1} b$

and the unique y is given by $x = A^H y$

$x = A^H (AA^H)^{-1} b$

$x = A^{-R} b$

Observation

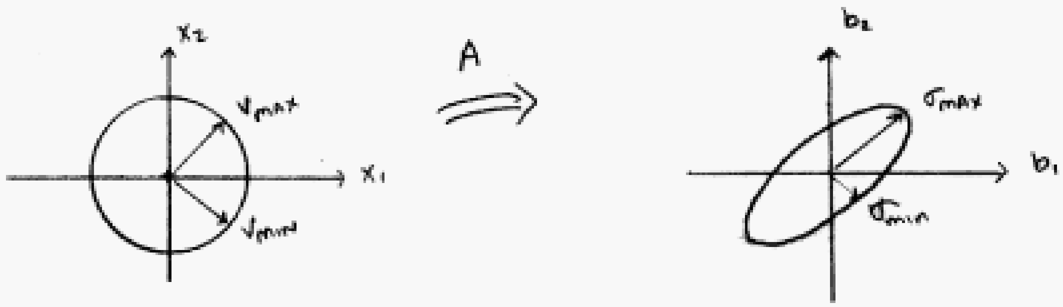
$$Ax = AA^H (AA^H)^{-1} b = b$$
$$x = A^H y = A^H (AA^H)^{-1} b$$
$$= A^H (AA^H)^{-1} A x$$

↑ projects vectors onto row space of A

SVD 's

Fact $\mathcal{R}(A) = \mathcal{R}(u_1)$

$$u = [u_1 \quad u_2]$$



$$A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Fact : $Av_i = \sigma_i u_i$

specifically :

$$A v_{max} = \sigma_{max} u_{max}$$ ← Left SV of A ...

↑ right singular vector of A corresponding to σ_{max}

direction of σ_{max} corresponds to $\underbrace{U_{max}}$

unit vector along major axis