- <u>observability</u>

  dual to controllability

<u>Analogies</u>:

<u>controllability</u>	<u>observability</u>
A	A
B	C — $p \times n$
$C[A,B] = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$	$O[A,C] = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}_{np \times n}$

(A,B,C,D) is observable $\Leftrightarrow$ (A,C) is observable

$\Leftrightarrow$ rank $O[A,C] = n = \#$ of columns

<u>PBH observability test</u>

(A,B,C,D) is observable $\Longleftrightarrow$ $\exists\ x\ (\neq 0)$ and $\lambda \in \mathbb{C}$ s.t.

$$\begin{bmatrix} C \\ \lambda I - A \end{bmatrix} x = 0$$

<u>Questions</u>

1) what does concept of observability attempt to address ?

2) where does PBH observability test come from ?

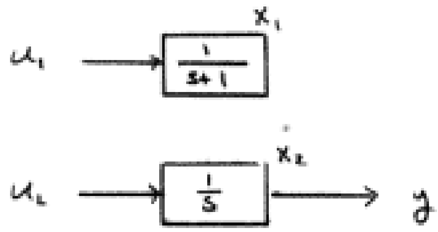
<u>Answer to 1</u> :

when can we estimate (reconstruct) uniquely the state x of a system given knowledge of y and u.

y = output

u = input

Example

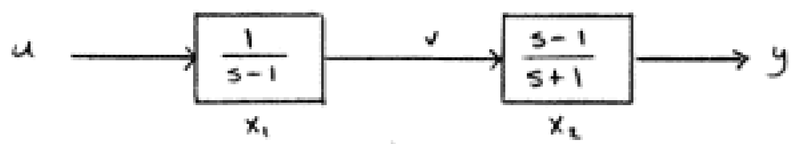


x_1 is controllable from u_1

x_1 is not observable from y

$s=-1$ mode is not observable from y.

Example



$$\frac{s-1}{s+1} = \frac{s+1-2}{s+1} = 1 + \frac{-2}{s+1}$$

State Equations

$$\dot{x}_1 = x_1 + u \qquad \dot{x}_2 = -x_2 - 2v$$
$$v = x_1 \qquad y = x_2 + v$$

Facts:

1) system, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is controllable from u

2) System is unobservable from y.
   - specifically, $s=1$ mode is unobservable from y

check:

$$\dot{x}_1 = x_1 + u$$
$$\dot{x}_2 = -x_2 - 2\overbrace{v}^{x_1}$$
$$y = x_2 + \overbrace{v}^{x_1}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0]u$$

$$\mathcal{C}[A,B] = [B \;\; AB] = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix} \Rightarrow \text{Rank} = 2 \text{ – controllable}$$

$$\mathcal{O}[A,C] = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \Rightarrow \text{Rank} = 1 \text{ – unobservable}$$

- <u>Why is it unobservable</u> ?

  $s = 1$ mode is unobservable

- <u>Why is $s=1$ mode unobservable from y</u>.

  pole-zero cancellation at system output

<u>PBH Test</u>

$$(sI - A)x = \begin{bmatrix} s-1 & 0 \\ 2 & s+1 \end{bmatrix} x \overset{s=1}{=} \begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix} x$$

$$x = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \begin{bmatrix} C \\ sI - A \end{bmatrix} x = 0 \quad \text{for } s = 1$$

$s = 1$ is an unobservable mode (from y)

<u>Why does PBH obs Test make sense</u> ?

$$y = Cx + Du$$

$$\hat{y} = \underbrace{y - Du}_{\text{known}} = \overset{\text{known}}{C}\underset{?}{x} \qquad C \text{ is not invertible}$$

<u>How do we determine x uniquely</u> ?

$$\hat{y} = Cx$$

$$\dot{\hat{y}} = C\dot{x} = CAx + CBu$$

$$\ddot{\hat{y}} = C\ddot{x} = CA\dot{x} + CB\dot{u}$$

$$= CA^2x + CBu + CB\dot{u}$$

Pattern

$$\hat{y} = Cx$$
$$\dot{\hat{y}} = CAx + CBu$$
$$\ddot{\hat{y}} = CA^2x + CBu + CB\dot{u}$$

stuff that I know $= \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \end{bmatrix} x$ useless

Back to PBH Test Origin !

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)} Bu(\tau)\,d\tau$$
$$y(t) = Cx(t) + Du(t)$$

don't know x_0

stuff that we know $= Ce^{At}x_0$ what is x_0 ?

Assume : A has n L.I. eigenvectors

$$A = X \Lambda X^{-1} = \sum_{i=1}^{n} \lambda_i x_i y_i^H$$
$$e^{At} = Xe^{\Lambda t}X^{-1} = \sum_{i=1}^{n} e^{\lambda_i t} x_i y_i^H$$

$$\text{stuff that we know} = C \Sigma e^{\lambda_i t} x_i \, y_i^H x_0$$
$$= \sum_{i=1}^{n} (y_i^H x_0)\, e^{\lambda_i t}\, Cx_i$$

If $Cx_k = 0$ for some k then λ_k is "unobservable" because k^{th} mode drops out in the above summation (can't be seen)

Suppose $Cx_5 = 0$ and $x_0 = x_5$

what does the above sum look like ?

Known stuff $= y_1^H x_0 e^{\lambda_1 t} C x_1 + \cdots 0 \cdots +$

5th term not present

All other terms are present

If $x_0 = x_5$

Note $y_i^H x_5 = 0$ for all $i \neq 5$

Known stuff $= 0$ (can't determine x_0 uniquely)