EEE 586 HW # 1 SOLUTIONS

Problem 1.6 Pick as states $x = [q_1; \dot{q}_1; q_2; \dot{q}_2]$. Then

$$\begin{pmatrix} 1 & & & \\ & M(x_1) & & \\ & & 1 & \\ & & & J \end{pmatrix} \frac{dx}{dt} = \begin{pmatrix} x_2 \\ -h(x_1, x_2) - K(x_1 - x_3) \\ x_4 \\ K(x_1 - x_3) + u \end{pmatrix} \Rightarrow \frac{dx}{dt} = \begin{pmatrix} x_2 \\ \frac{-h(x_1,x_2) - K(x_1 - x_3)}{M(x_1)} \\ x_4 \\ \frac{K(x_1 - x_3)}{J} + \frac{1}{J} u \end{pmatrix}$$

Problem 1.15 Differentiating the $L\cos\theta$ and $L\sin\theta$ terms we get

$$\begin{aligned} m\ddot{y} + mL(\ddot{\theta}\cos\theta - \dot{\theta}^2 \sin\theta) &= H \\ -mL(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta) &= V - mg \end{aligned}$$

Solving for H, V and substituting, we obtain

$$\begin{aligned} I\ddot{\theta} &= mgL\sin\theta - mL^2\ddot{\theta} - mL\ddot{y}\cos\theta \\ M\ddot{y} &= F - m\ddot{y} - mL\ddot{\theta}\cos\theta + mL\dot{\theta}^2\sin\theta - k\dot{y} \end{aligned}$$

Collecting the second derivatives in the left-hand side,

$$\begin{pmatrix} I + mL^2 & mL\cos\theta \\ mL\cos\theta & M + m \end{pmatrix} \begin{pmatrix} \ddot{\theta} \\ \ddot{y} \end{pmatrix} = \begin{pmatrix} mgL\sin\theta \\ F + mL\dot{\theta}^2\sin\theta - k\dot{y} \end{pmatrix}$$

Now, the inverse of the left-hand side matrix is

$$\begin{pmatrix} M + m & -mL\cos\theta \\ -mL\cos\theta & I + mL^2 \end{pmatrix} \frac{1}{\Delta(\theta)}$$

where $\Delta(\theta)$ is the determinant of the matrix, i.e.,

$$\Delta(\theta) = (I + mL^2)(m + M) - m^2L^2\cos^2\theta = Im + IM + m^2L^2\sin^2\theta + MmL^2$$

which is always positive and satisfies the stated inequality ($> (I + mL^2)M + mI$).

Problem 2.2 1. At an equilibrium $\dot{x} = 0$. The possible solutions are $(0, 0)$, $(2, 0)$, $(-2, 0)$. To determine the type, we look at the linearization:

$$A \triangleq \frac{\partial f}{\partial x} = \begin{pmatrix} 0 & 1 \\ -1 + \frac{5x_1^4}{16} & -1 \end{pmatrix}$$

whose eigenvalues at the three equilibria are: $\{-0.5 \pm j0.866\}$, $\{1.56, -2.56\}$, $\{1.56, -2.56\}$, respectively. Thus, the origin is a stable focus and the other two equilibria are saddle points.

2. At an equilibrium $\dot{x} = 0$. The possible solutions are $(0, 0)$, $(1, 2)$, $(-1, 2)$. To determine the type, we look at the linearization:

$$A \triangleq \frac{\partial f}{\partial x} = \begin{pmatrix} 2 - x_2 & -x_1 \\ 4x_1 & -1 \end{pmatrix}$$

whose eigenvalues at the three equilibria are: $\{2, -1\}$, $\{-0.5 \pm j1.94\}$, $\{-0.5 \pm j1.94\}$, respectively. Thus, the origin is a saddle point and the other two equilibria are stable foci.

3. At an equilibrium $\dot{x} = 0$ so $x_2 = 0$ implying that $\psi(x_1) = 0$. The only solution is $(0, 0)$. To determine the type, we look at the linearization:

$$A \triangleq \frac{\partial f}{\partial x} = \begin{pmatrix} 0 & 1 \\ -\frac{\partial\psi}{\partial x_1} & -1 + \frac{\partial\psi}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -0.5 & -1 + 0.5 \end{pmatrix}$$

whose eigenvalues are $\{-0.25 \pm j0.66\}$. Thus, the origin is a stable focus.

Problem 2.17 3. There is only one equilibrium at the orign and it is an unstable focus. Let $V = \frac{1}{2}x_1^2+x_2^2+x_1x_2$. The last term introduces $-x_1^2$ in the derivative and the coefficients are chosen to eliminate some of the cross terms (x_1x_2), while keeping V positive definite. Its derivative along the trajectories of the system is $\dot{V} = -x_1^2 + 3x_2^2 - 2x_2^2(x_1 + 2x_2)^2$. To apply the P-B Criterion, we need $\dot{V} < 0$ for large enough $|x|$. A quick way to verify this is to use a mesh plot in MATLAB. The analytical version is a bit more tedious.

First, redefine variables as $z_1 = x_1$ and $z_2 = x_1 + 2x_2$. Then $\dot{V} = -z_1^2 - \frac{1}{2}(z_2 - z_1)^2(z_2^2 - 3/2)$. Notice that $\dot{V} < 0$ in the set $M_1 = \{z : z_2^2 > 3/2\}$. It suffices to show that $\dot{V} < 0$ in the set $M_2 = \{z : z_2^2 < 3/2\} \cap \{z : |z| > r\}$ for large enough r. In M_2, $z_2^2 < 3/2$ so $\dot{V} \leq -z_1^2 + \frac{3}{4}(z_2 - z_1)^2$. Moreover, $|z| > r \Rightarrow z_1^2 > r^2 - z_2^2 > r^2 - 3/2$. So,

$$\dot{V} \leq -\frac{1}{4}z_1^2 + \frac{3}{4}z_2^2 - \frac{3}{2}z_2z_1 \leq -\frac{1}{4}z_1^2 + \frac{9}{8} + 1.84|z_1|$$

The roots of the quadratic are $7.92, -0.567$, so $\dot{V} < 0$ for $|z_1| > r = 8$. Notice that $\dot{V} < 0$ in $M_1 \cup M_2$ and $(M_1 \cup M_2)^c = \{z : z_2^2 \leq 3/2 \ \& \ |z| \leq r\}$ which is a compact set. Define the set M as the smallest level set $\{V \leq c\}$ containing $(M_1 \cup M_2)^c$. This implies that $\dot{V} < 0$ in M and trajectories of the system that begin in M stay in M. Hence, by the P-B Criterion there exists a periodic orbit in M.

In this case, the choice of our coordinates is such that $V(x) = z^\top z/4$, so the set M is easy to compute: $M = \{x : V(x) < 16\}$. In general, its computation involves an optimization problem that may or may not be computable analytically. In the attached figures, notice the conservatism in the estimate of the set M containing the limit cycle. It goes without saying that other functions V may produce different or better results.

Problem 2.18 a. For the given V, $\dot{V} = x_2\dot{x}_2+g(x_1)\dot{x}_1 = 0$. Hence, V is constant along the system trajectories.

b. For small c, V is a positive definite function. Therefore, $V = c$ cannot contain the origin. And since it must be an increasing function of the state norm for small c, it cannot contain points at infinity. So $V = c$ defines a closed curve that does not contain an equilibrium and trajectories starting on it stay on the curve (from Part a.). Thus, the trajectories starting close to the origin, are periodic orbits.

Problem 2.20 2. Rewriting the ODEs in terms of $(1 - x_1^2 - x_2^2)$ we find that there is one equilibrium at the origin and a continuum of non-isolated equilibria on the unit circle. This expression motivates the consideration of polar coordinates, and in particular the evolution of the radius $r^2 = x_1^2 + x_2^2$. For this, $\dot{r} = -r(1 - r^2)$. This is a scalar differential equation and can be analyzed directly. For $r < 1$, the trajectories converge to the origin. For $r > 1$ trajectories diverge to infinity. Hence, there are no limit cycles.