EEE 586 HW # 2 SOLUTIONS

Problem 3.1

6. $f(x) = \tan x$ is not continuous at $\pi/2$ but it is at 0. It is only locally continuously differentiable. It is locally Lipschitz around 0.

8. $f(x) = [-x_1 + a|x_2|; -(a+b)x_1 + bx_1^2 - x_1x_2]$ is not continuously differentiable at 0 (unless a=0). It is continuous everywhere and locally Lipschitz. It is not globally Lipschitz because of the x_1^2 and x_1x_2 terms.

Problem 3.3

Let f_1, f_2 be locally Lipschitz: $|f_i(x) - f_i(y)| \le k_i|x-y|$, for all $x, y \in D$.

Then, $|f_1(x) + f_2(x) - f_1(y) - f_2(y)| \le |f_1(x) - f_1(y)| + |f_2(x) - f_2(y)| \le (k_1 + k_2)|x-y|$, in D. Hence $f_1 + f_2$ is Lipschitz.

Also, $|f_1(x)f_2(x) - f_1(y)f_2(y)| \le |f_1(x)f_2(x) - f_1(y)f_2(y) \pm f_1(x)f_2(y)| \le |f_1(x)||f_2(x) - f_2(y)| + |f_2(y)||f_1(x) - f_1(y)| \le K|x-y|$, in D, as long as f_1, f_2 are bounded in D. That is the case, for example, if f_i are continuous and D is compact. In general, f_1f_2 is locally Lipschitz

Finally, $|f_2 \circ f_1(x) - f_2 \circ f_1(y)| = |f_2[f_1(x)] - f_2[f_1(y)]| \le k_2|f_1(x) - f_1(y)| \le k_2k_1|x-y|$. So $f_1 \circ f_2$ is Lipschitz in the same domain.

Problem 3.4 $\|Kx\|$ is Lipschitz in x and, using 3.3, $g(x)\|Kx\|$ and $g(x)K(x)$ are Lipschitz on any compact set. (Note: continuity on compact sets implies boundedness.)

When $g(x)\|Kx\| > \mu$, then $1/\|Kx\| < g(x)/\mu$ so it is bounded on compact sets. That means that $Kx/\|Kx\|$ is Lipschitz.

It remains to show continuity at the boundary. Consider $g(x)\|Kx\| = \mu - \eta$, $g(y)\|Ky\| = \mu + \eta$, for some small $\eta > 0$. Then $g(x)Kx/\mu - Ky/\|Ky\| = g(x)\|Kx\|Kx/\mu\|Kx\| - Ky/\|Ky\| = (1 - \eta/\mu)Kx/\|Kx\| - Ky/\|Ky\|$. Since $Kx/\|Kx\|$ is Lipschitz, the norm of the last expression is less than $\eta/\mu + L\|x-y\|$. Since this can be made arbitrarily small (for sufficiently small $x - y$ and η), we can establish the continuity at the boundary.

So f is locally Lipschitz on any compact set.

Problem 3.23

Set $g(\sigma) = f(\sigma x),\ 0 \le \sigma \le 1$. Since D is convex, $\sigma x \in D$ for $0 \le \sigma \le 1$. Then,

$$g'(\sigma) = f'(\sigma x)\frac{d\sigma x}{d\sigma} = f'(\sigma x)x$$

$$f(x) = f(x) - f(0) = g(1) - g(0) = \int_0^1 g'(\sigma)\, d\sigma = \int_0^1 f'(\sigma x)\, d\sigma\, x$$

Problem 4.8 $V = \frac{x_1^2}{u} + x_2^2$, $u = 1 + x_1^2$ is locally pdf and not RU. Taking the derivative, $\dot{V} = \frac{-12x_1^2}{u^4} - \frac{4x_2^2}{u^2} < 0$, implying local AS.

Considering the hyperbola $x_2 = 2/(x_1 - \sqrt(2))$, rewrite the equation as $g(x) = 0$ and form the gradient $\nabla g = [2/(x_1 - sqrt(2)), 1]$. Then, we want to show that the inner product $< f, \nabla g >> 0$ on the hyperbola. While the substitution is straightforward, the expressions are quite long. Here, a Matlab evaluation is helpful to show that the inner product is positive meaning that the vector field always points to the right of the hyperbola.

The last observation implies that initial conditions starting to the right of the hyperbola remain there. Hence 0 cannot be GAS.

Problem 4.10

Using the suggested expression,

$$x^\top Pf + f^\top Px = x^\top \left(\int_0^1 \left[P\frac{\partial f(\sigma x)}{\partial x} + \frac{\partial f(\sigma x)}{\partial x}^\top P \right] d\sigma \right) x \le -x^\top x$$

Next, $V = f^\top Pf \ge 0$, for any f. It is PD iff $V = 0 \Leftrightarrow x = 0$. This is equivalent to $f = 0 \Leftrightarrow x = 0$, i.e., 0 is a unique equilibrium of $\dot{x} = f(x)$. We show this by contradiction. Suppose that there is $q \ne 0$ such that $f(q) = 0$. Then, from Part 1, $q^\top Pf(q) + f(q)^\top Pq \le -q^\top q$. But the left hand side is 0 so $q = 0$ which is a contradiction.

Next, we need to show that V is RU. That is, as $|x| \to \infty$, so must f. Again, suppose that it is not so and $|f(x)| < c$ as x grows. From Part 1, $\frac{x^\top P f}{x^\top x} \leq -\frac{1}{2}$. But if f were bounded, then the fraction satisfies $\frac{x^\top P f}{x^\top x} \leq \frac{|P|c}{|x|} \to 0$ which is a contradiction.

Thus, we have that V is PD and RU. Furthermore, $\dot{V} \leq f^\top (P\frac{\partial f}{\partial x} + \frac{\partial f}{\partial x}^\top P) f \leq -f^\top I f < 0$. So 0 is GAS.

References

[1] Braun, M., *Differential equations and their applications*, Springer-Verlag 1983.

[2] Hale, J. K., *Ordinary differential equations*, Krieger Publishing Company 1980.