EEE 586 HW # 3 SOLUTIONS

Problem 4.13

1. Let $V = \frac{1}{2}(x_1^2 - x_2^2)$. Then

$$\begin{aligned}
\dot{V} &= x_1^4 + x_1^3 x_2 + x_2^2 - x_2^3 - x_1 x_2^2 + x_1^3 x_2 \\
&= x_1^4 + \tfrac{1}{2}x_2^2 + 2x_1(x_1^2 x_2) + x_2^2(\tfrac{1}{2} - x_2 - x_1) \\
&\geq x_1^4 + \tfrac{1}{2}x_2^2 + 2x_1(x_1^2 x_2) \quad \text{for } |x| \leq \tfrac{1}{4} \\
&\geq (x_1^2 - \tfrac{1}{\sqrt{2}}|x_2|)^2 + \tfrac{2}{\sqrt{2}}x_1^2|x_2| - 2|x_1|(x_1^2|x_2|) \\
&\geq 0 \quad \text{for } |x| \leq \tfrac{1}{\sqrt{2}}
\end{aligned}$$

Combining the two conditions, we get that $\dot{V} \geq 0$ whenever $|x| \leq \frac{1}{4}$. Furthermore, $V > 0$ in $U = \{x : |x_1| > |x_2|\}$, that has zero on its boundary (we write $0 \in \partial U$). Hence, 0 is an unstable equilibrium.

2. (*corrected*) This solution is based on more elementary observations:

Consider the set $-x_1^3 + x_2 >= 0$. On its boundary in the 1st quadrant, $\dot{x}_2 \geq 0$. Also, consider the set $x_1^6 - x_2^3 >= 0$. On its boundary in the 1st quadrant, $\dot{x}_1 \geq 0$. The intersection of these two sets is therefore a positively invariant set. Since both elements of $\dot{x}$ are positive, any trajectory starting inside it moves towards higher values (the equilibrium at [1,1]). Since the same is true for trajectories starting arbitrarily close to 0, we conclude that 0 is unstable.

Problem 4.15

a. The equilibrium point has $x_2 = 0$, $h_1(x_1) - h_2(x_3) = 0$, $x_3 = x_2 = 0$, hence $h_1(x_1) = 0$, hence $x_1 = 0$. The origin is the unique equilibrium.

b. Clearly. $V(x) \geq 0$. To show PD, we need to show that $V(x) = 0 \Rightarrow x = 0$. This follows since $y h_i(y) > 0$, which implies that $\int_0^z h_i(y)dy$ vanishes only at $z = 0$.

c. $\dot{V} = h_1(x_1)x_2 + x_2[-h_1(x_1) - x_2 - h_2(x_3)] + h_2(x_3)(x_2 - x_3) = -x_2^2 - x_3 h_2(x_3) \leq 0$ We now apply the invariance theorem: $\dot{V} = 0$ when $x_2 = x_3 = 0$. The second state equation yields $0 = -h_1(x_1)$ so $x_1 = 0$. Hence, the origin is AS.

d. To show asymptotic stability, we need to show that V is radially unbounded. That is for $\|x\| \to \infty$, we should have $V \to \infty$. The former implies that at least one of $x_i \to \infty$. Since V is the sum of three decoupled terms, we should have that the corresponding term also grows unbounded. So, $\int_0^z h_i(y)dy \to \infty$ as $|z| \to \infty$ (h should not decay faster than $1/z$).

Problem 4.16

Let $V = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$. Then

$$\dot{V} = x_1^3 x_2 + x_2(-x_1^3 - x_2^3) = -x_2^4 \leq 0$$

This implies that 0 is uniformly stable. Next, using La Salle, $\dot{V} = 0 \Rightarrow x_2 = 0$. In the set where $x_2 \equiv 0$, we have $\dot{x}_2 = 0$ and therefore, $x_1 = 0$. Hence, the only trajectory in this set is 0, so the equilibrium is UAS. Furthermore, since V is RU, 0 is GUAS.

Problem 4.25

Controllability of (A, B) implies that any initial state can be transfered to the origin in time $\tau > 0$ so the finite-interval $[0, \tau]$ gramian should be nonsingular, and therefore positive definite. (See any text on Linear System Theory for this.) Hence, its inverse is also positive definite.

Viewing the Lyapunov equation as the solution of a differential equation, we get

$$AW + WA^\top = -\int_0^\tau \frac{d}{dt}\left\{e^{-At}BB^\top e^{-A^\top t}\right\} dt = BB^\top - e^{-A\tau}BB^\top e^{-A^\top \tau}$$

Hence,

$$(A - BK)W + W(A - BK)^\top = AW + WA^\top - 2BB^\top = -BB^\top - e^{-A\tau}BB^\top e^{-A^\top \tau}$$

Letting $V(x) = x^\top W^{-1} x$, we have

$$\dot{V} = x^\top W^{-1}[(A-BK)W + W(A-BK)^\top]W^{-1}x = -x^\top W^{-1}[BB^\top + e^{-A\tau}BB^\top e^{-A^\top\tau}]W^{-1}x \leq 0$$

So the origin is a stable equilibrium.

For asymptotic stability, we need to show that no eigenvalue λ of $A-BK$ can have a zero real part. Suppose there is one and let ν be the corresponding left eigenvector. Then $\nu^*(A-BK) = \lambda\nu^*$. Taking the quadratic of $W^{-1}\nu$ with the lyapunov derivative we get

$$\nu^*[(A-BK)W + W(A-BK)^\top]\nu = \lambda\nu^*W\nu + \lambda^*\nu^*W\nu = 2Re[\lambda]\nu^*W\nu = -\nu^*[BB^\top + e^{-A\tau}BB^\top e^{-A^\top\tau}]\nu$$

But $Re[\lambda] = 0$ so $\nu^*B = 0$ which implies that $(A-BK, B)$ is not controllable. Since controllability is invariant under state feedback, this is a contradiction.

Problem 4.54

1. $\dot{x} = -(1+u)x^3$. Let $u = -2$, $x(0) > 0$. Then x diverges. The system is not I2S stable.
2. $\dot{x} = -(1+u)x^3 - x^5$. Let $V = x^2/2$. Then $\dot{V} = -x^6 - x^4 - ux^4 \leq -x^4$ for $|x| \geq \sqrt{|u|}$. The system is I2S stable.
3. $\dot{x} = -x + x^2 u$. Let $u = 1$, $x(0) > 1$. Then x diverges. The system is not I2S stable.
4. $\dot{x} = x - x^3 + u$. The 0 equilibrium for $u = 0$ is unstable and, therefore, the system is not I2S stable. (But it is bounded with bound $|x| \sim \max(1, \sqrt{|u|})$. The constant 1 is to account for the fact that $|x|$ approaches 1 as u approaches 0.)

Problem 4.56

The system $\dot{x}_1 = -x_1^3 + x_2$ with x_2 as an input is I2S stable. The system $\dot{x}_2 = -x_2^3$ is has the origin as a UAS equilibrium. By Lemma 4.7, the origin of the combined system is GUAS.