EEE 586 HW # 4 SOLUTIONS

Problem 5.10.2 Take $V = \frac{1}{2}x^2$. Then $\dot{V} = -x^4 + ux^4 - x^6 \le -x^4$, for $|x| \ge \sqrt{|u|}$. Therefore, the system is I2S stable. From Thm. 5.3 it follows that the system is L_∞ stable. For finite gain stability using Thm. 5.1, we need the unforced system to be exponentially stable, which is not true. But locally, for $|u| < 1$ the *forced* system is UAS, hence $|y(t)| \le \beta(x_0, t) + |u(t)|$, so the system is small-signal, finite-gain L_∞ stable.

Problem 5.10.4 Take $V = \frac{1}{2}x^2$. Then $\dot{V} = -x^2 - x^4 + ux^3 \le -x^2$, for $|x| \ge |u|$. Therefore, the system is I2S stable. Since $|y| \le |x|$, from Thm. 5.3 it follows that the system is L_∞ stable. We do not expect finite gain stability since $\partial f/\partial u$ is not bounded.

Problem 5.11.6 For the system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1^3 - x_2 + u, \quad y = x_2$$

the linearization is not ES, hence Thm. 5.1 is not applicable. But the system is I2S stable (Problem 4.55.3), and Thm. 5.3 is applicable. Hence it is L_∞ stable.

Problem 5.15.3 For the system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_1 - sat(2x_1 + x_2) + u, \quad y = x_1$$

we have a linearization matrix with eigenvalues in the LHP. Corollary 5.1 is applicable, showing s.s.-$\mathcal{L}_p$-stability with finite gain. By defining the s.s. region as $\{x : |2x_1 + x_2| < 1\}$, the s.s. $\mathcal{L}_2$ gain is the $\mathcal{H}_\infty$ norm of the linearization transfer function: $G(s) = \frac{1}{s^2+s+1}$. Its magnitude on the $j\omega$-axis satisfies $|G(j\omega)|^2 = \frac{1}{(-\omega^2+1)^2+(\omega)^2}$. Hence, the maximum $|G(j\omega)|$ is $\frac{2}{\sqrt{3}} \left(= \frac{1}{2\zeta\sqrt{1-\zeta^2}} \right)$. (Note: Depending on the initial condition, u has to be small enough so that the system never leaves the s.s. region. This is always possible since the linearized system is ES and, hence, the map $u \mapsto x$ has finite Ł_∞ gain.)

We do not expect any global result since for large x, the saturation nonlinearity prevents the stabilization of the system. As an example, consider the case where $x_1 > 0, x_2 > 0, u > 1$. Then $\dot{x}_2 > 0$, meaning that x_2 stays positive and increases. Also, $\dot{x}_1 > 0$, meaning x_1 stays positive and increases.

- **5.17**

$$\begin{aligned}
\frac{\partial V}{\partial x}f + \frac{\partial V}{\partial x}Gu &= \frac{\partial V}{\partial x}f + \frac{\partial V}{\partial x}Gu - \frac{1}{2}(L+Wu)^T(L+Wu) + \frac{1}{2}(L+Wu)^T(L+Wu) \\
&= -\frac{1}{2}(L+Wu)^T(L+Wu) + \frac{\partial V}{\partial x}f + \frac{\partial V}{\partial x}Gu + \frac{1}{2}L^TL + L^TWu + \frac{1}{2}u^TW^TWu \\
&= -\frac{1}{2}(L+Wu)^T(L+Wu) + \left\{ \frac{\partial V}{\partial x}f + \frac{1}{2}L^TL + \frac{1}{2}h^Th \right\} \\
&\quad -\frac{1}{2}h^Th + \frac{\partial V}{\partial x}Gu - h^TJu - \frac{\partial V}{\partial x}Gu + \frac{1}{2}u^T(\gamma^2I - J^TJ)u \\
&= -\frac{1}{2}(L+Wu)^T(L+Wu) + \mathcal{H} - \frac{1}{2}h^Th - h^TJu + \frac{1}{2}\gamma^2u^Tu - \frac{1}{2}u^TJ^TJu \\
&= -\frac{1}{2}(L+Wu)^T(L+Wu) + \mathcal{H} + \frac{1}{2}\gamma^2u^Tu - \frac{1}{2}y^Ty
\end{aligned}$$

$\mathcal{H} \le 0$ implies

$$\frac{\partial V}{\partial x}f + \frac{\partial V}{\partial x}Gu \le \frac{1}{2}\gamma^2u^Tu - \frac{1}{2}y^Ty$$

From this point on, proceed as in the proof of Theorem 5.5 (starting from (5.29)).