Chapter 3

The LTV Plant

3.1 Introduction

In this chapter we define the class of LTV plants for which we subsequently formulate and study certain control problems in the progressively harder cases of complete and incomplete a priori knowledge of parameter variations.

The choice of the control objective, for which the control problem has a meaningful solution, depends of course on the underlying structure of the plant and, conversely, for the control problem to be well-posed the plant should satisfy certain assumptions depending on the selected control objective. An issue of particular interest in our study is the control of plants whose TV parameters are only partially known. In such a case, one may attempt to meet the control objective by combining a parameter estimator or adaptive law with a particular controller structure. The underlying intuitive idea behind this approach is that the parameter estimator uses I/O information to estimate the unknown parameters on-line while the updated parameter estimates are used in the calculation of the control input signal.

Inherent in this approach is the concept of plant parametrization, that is, a description of the plant I/O operator in terms of some parameters. However, not all possible parametrizations of LTV I/O operators are convenient for the identification of the I/O operator via a parameter estimation algorithm. For example, if we consider an arbitrary state-space description of an LTV system $\dot{x} = Ax + bu$, $y = c^{\top}x$ and its natural parametrization in terms of the triple $[A, b, c]$, it may not be possible to design an estimator which determines the parameters $[A, b, c]$ uniquely from I/O information. In order to achieve such an objective, we need to impose certain conditions on the values and structure of the triple $[A, b, c]$.

In this chapter we provide the conditions ensuring that a general class of LTV plants admits a convenient parametrization for estimation and control purposes. We begin with Section 3.2 where we discuss the analytically simpler case of plant representations with smooth parameter variations. In Section 3.3 we consider LTV plants with non-smooth, possibly discontinuous parameter variations. Finally, convenient for parameter estimation plant parametrizations are studied in Section 3.4, for both smooth and non-smooth parameter variations.

3.2 Smooth Parameter Variations

Consider a SISO LTV plant described by the differential equation

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + b(t)u_p(t) \; ; \quad x(t_0) = x_0 \\ y_p(t) &= c^{\top}(t)x(t) \end{aligned} \tag{3.1}$$

where $t_0 \in \mathbf{R}_+$, $(u_p, y_p)(t) \in \mathbf{R} \times \mathbf{R}$ and $x(t) \in \mathbf{R}^n$, satisfying the following assumptions:

3.1 Assumption: *$A(t), b(t), c(t)$ are smooth, UB functions of time with UB derivatives.* ∎

3.2 Assumption: *The triple $[A(t), b(t), c(t)]$ is strongly controllable and observable in $[t_0, \infty)$.* ∎

3.3 Assumption: *The order of the plant, denoted by n, is constant and finite.* ∎

Under these assumptions, Lemma 2.32 ensures that the plant I/O map is described by a topologically equivalent state-space representation which is in either the controllable canonical form, i.e.,

$$\dot{x}_c = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & 1 \\ -a_n(t) & -a_{n-1}(t) & -a_{n-2}(t) & \dots & -a_1(t) \end{bmatrix} x_c + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u_p$$

$$y_p = [b_{n-1}(t),\, b_{n-2}(t), \dots, b_0(t)] x_c \tag{3.2}$$

or the observable canonical form, i.e.,

$$\dot{x}_o = \begin{bmatrix} -a'_1(t) & 1 & 0 & \dots & 0 \\ -a'_2(t) & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & 1 \\ -a'_n(t) & 0 & 0 & \dots & 0 \end{bmatrix} x_o + \begin{bmatrix} b'_0(t) \\ b'_1(t) \\ \vdots \\ b'_{n-1}(t) \end{bmatrix} u_p$$

$$y_p = [1, 0, \dots, 0] x_o \tag{3.3}$$

We refer to (3.2) as the P_R form and to (3.3) as the P_L form of the plant. As noted in Example 2.22, a plant in the P_R form has I/O operator

$$y_p = G_p^R(s,t)[u_p] \ ; \quad G_p^R(s,t) = N_p(s,t) D_p^{-1}(s,t) \tag{3.4}$$

where

$$D_p(s,t) = s^n + a_1(t) s^{n-1} + \dots + a_n(t)$$

$$N_p(s,t) = b_0(t) s^{n-1} + b_1(t) s^{n-2} + \dots + b_{n-1}(t)$$

are PDO's in the left form with smooth, UB coefficients. Furthermore, due to Assumption 3.2 and from Corollary (2.34), $D_p(s,t)$, $N_p(s,t)$ are strongly right coprime PDO's in $[t_0, \infty)$.

Similarly, using Example 2.23 we obtain that a plant in the P_L form has I/O operator

$$y_p = G_p^L[u_p] \ ; \quad G_p^L(s,t) = D_p^{-1}(s,t) N_p(s,t) \tag{3.5}$$

where

$$D_p(s,t) = s^n + s^{n-1} a'_1(t) + \dots + a'_n(t)$$

$$N_p(s,t) = s^{n-1} b'_0(t) + s^{n-2} b'_1(t) + \dots + b'_{n-1}(t)$$

are PDO's in the right form with smooth, UB coefficients. Also, due to Assumption 3.2 and from Corollary (2.34), $D_p(s,t)$, $N_p(s,t)$ are strongly left coprime PDO's in $[t_0, \infty)$.

For simplicity, we use the same notation for the PDO's and PIO's of the plant I/O operator in either the P_R or P_L form; their meaning is clear from the context. Moreover, we denote the coefficients of the plant PDO and PIO ($a_i(t), b_i(t)$ or $a'_i(t), b'_i(t)$) by the vector $\Theta_p(t)$.

Further, the *relative degree* of the plant G_p^R or G_p^L

$$n^* \triangleq \deg[D_p(s,t)] - \deg[N_p(s,t)]$$

is equal to one on any interval where $b_0(t) \neq 0$. If, for some constant integer $m \leq n-1$ we have that

$$b_0(t) = b_1(t) = \cdots = b_{n-m-2}(t) = 0 \ ; \quad b_{n-m-1} \neq 0$$

for all $t \in [t_0 \infty)$, then the relative degree of the plant is $n-m$. In this case, the I/O description (3.4) can be written as

$$y_p = G_p^R[u_p] \ ; \quad G_p^R(s,t) = k_p(t) N_p(s,t) D_p^{-1}(s,t) \tag{3.6}$$

where, now, $N_p(s,t)$ is a monic PDO of degree m and with coefficients $\bar{b}_i(t) = b_i(t)/b_{n-m-1}(t)$ and $k_p(t) = b_{n-m-1}(t)$ is the so called *high-frequency gain*. Similarly, if the plant has relative degree $n-m$, the I/O description (3.5) can be written as

$$y_p = G_p^L[u_p] \ ; \quad G_p^L(s,t) = D_p^{-1}(s,t) N_p(s,t) k_p(t) \tag{3.7}$$

where $N_p(s,t)$ is a monic PDO of degree m and with coefficients $\bar{b}'_i(t) = b'_i(t)/b'_{n-m-1}(t)$ and $k_p(t) = b'_{n-m-1}(t)$.

The above assumptions capture some of the essential properties that are required for the well-posedness of the control problem. To clarify the meaning and give an interpretation of the assumptions we invoke the results of Chapter 2 to make the following observations.

In the plant representation (3.1) we have tacitly assumed that the state $x(t)$ includes the physical quantities, e.g., voltage, displacement, temperature etc., whose properties, such as continuity, boundedness etc., are of interest. Since in the LTV case algebraically equivalent systems do not necessarily share the same internal stability properties, the properties of the state $x(t)$ may not be deduced, in general, from the properties of the I/O operator of the plant. For this purpose, Assumption 3.2 introduces a notion of minimality and ensures that the plant is uniformly controllable and observable. Consequently, all states can be accessed with bounded inputs and no state can grow unbounded without being observed at the output.

Assumption 3.2 also guarantees that the plant can be brought to any of the canonical forms via a Lyapunov transformation and hence described by an I/O operator, factorized in terms of PDO's with smooth, UB coefficients (eqns. (3.4)–(3.7)). In addition, Corollary 2.34 shows that these PDO's are strongly (right or left) coprime.

3.3 Non-Smooth Parameter Variations

Among the critical assumptions about the plant, discussed in the previous section, were the smoothness of the plant parameters and the strong controllability/observability of the plant. Despite their generality, such assumptions exclude an important class of LTV plants with discontinuous or non-differentiable parameters.[1] In addition to the discontinuities, it may be possible that our assumptions are violated inside 'short' time intervals, e.g., during a transition between two different modes of operation of the plant. In such cases of non-smooth parameter variations one of the major points of concern, particularly for continuous time systems, is that it may not be possible at all to describe the LTV system by a convenient PDO/PIO factorization and/or perform the necessary PDO operations. The reason is that parameter smoothness is a property associated with the physical variables that affect the system behavior. Such variables are not necessarily related in a simple way with the parameters of a canonical form. For example, consider the PDO's $[s+a(t)]$ and $[s+b(t)]$ where $a(t), b(t)$ are discontinuous functions. Any attempt to express the product $[s+a(t)][s+b(t)]$ as a single PDO of degree two would cause the appearance of delta distributions in its coefficients.

[1] Although it is straightforward —and tedious— to derive the exact number of required differentiations, such calculations are omitted since they depend critically on the assumed state-space representation of the plant.

In our formulation we avoid any unnecessary further complications caused by such a description, by considering plants in the general state-sace representation (3.1). In order to handle discontinuous as well as non-differentiable system parameters we decompose, without loss of generality, the representation (3.1) as

$$\begin{aligned} \dot{x} &= A_o(t)x + b_o(t)u_p + \tilde{A}(t)x + \tilde{b}(t)u_p \\ y_p &= c_o^\top(t)x + \tilde{c}^\top(t)x \end{aligned} \tag{3.8}$$

where $A_o(t)$, $b_o(t)$, $c_o(t)$ are referred to as the 'nominal' part of the plant and $\tilde{A}(t)$, $\tilde{b}(t)$, $\tilde{c}(t)$ as the 'perturbation' part of the plant. Thus, instead of Assumptions 3.1–3.3, the LTV plant (3.8) is assumed to satisfy:

3.4 Assumption: *The entries of $A_o(t), b_o(t), c_o(t)$ are piecewise smooth, UB functions of time and the entries of $\tilde{A}(t)$, $\tilde{b}(t)$, $\tilde{c}(t)$ are piecewise continuous,*[2] *UB functions of time, satisfying*

$$\int_{t_0}^{t_0+T} \|\tilde{\cdot}\|^2 \leq C + \mu' T$$

where $C, \mu' \geq 0$ are some constants, for all $t_0 \geq 0$ and $T \geq 0$.[3] *Furthermore, let us denote by t_j, $j = 1, \ldots, \infty$, the points of discontinuity of $A_o(t)$, $b_o(t)$, $c_o(t)$ where $\{t_j\}_1^\infty$ is a strictly increasing sequence $\in \mathbf{R}_+$ with $t_j \to \infty$ as $j \to \infty$.*[4] *That is, $A_o(t), b_o(t), c_o(t)$ are smooth for all t except t_j.* ∎

3.5 Assumption: *The triple $[A_o(t), b_o(t), c_o(t)]$ satisfies Assumptions 3.1–3.3 inside each interval (t_j, t_{j+1}), $j \in \mathbf{N}$, uniformly in j.* ∎

3.6 Assumption: *There exist constants C, ν such that in any interval $(t_0, t_0 + T)$ the number of discontinuities n_I of the nominal plant parameters satisfies*

$$n_I \leq C + \nu T$$

$\forall t_0 \geq 0$, $\forall T \geq 0$. ∎

In other words, we consider LTV plants which are perturbations of 'well-behaved' plants, i.e., plants with smooth parameters satisfying our strong controllability and observability assumption as stated in the previous section. Such perturbations can have the form of a small-in-the-mean non-differentiable part, or infrequent discontinuities (jumps) added to the nominal parameters.

Of course, given a general non-smooth state-space representation of a plant, its decomposition into a nominal and a perturbation part is not unique and can be performed in several different ways. For example, a discontinuity in a parameter (or its derivative) can be included directly in the nominal part of the plant or a smooth approximation can be considered as a nominal part and the difference as a perturbation. Needless to say, although different representations of the same plant affect the conservatism in estimating regions of stability, the final result is qualitatively the same.

We must emphasize at this point that for the class of plants described by (3.8) to be a non-trivial extension of the smooth parameter case, we must allow for discontinuities in the nominal parameters. This is necessary in order to admit plants whose parameters cross a controllability/observability boundary. In such a case, the nominal part of the state-space description should either contain a jump or a loss in strong

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[2] Without loss of generality, both the nominal and perturbation part are assumed to be continuous from the right.

[3] Similar results can be obtained if $\int_{t_0}^{t_0+T} \|\tilde{\cdot}\| \leq C + \mu' T$ holds.

[4] If the number of discontinuities is finite, we may consider a sequence $\{t_j\}$ padded with points at infinity.

controllability/observability for a short time interval. As a simple example to illustrate this concept, let us consider the plant

$$\dot{x} = b(t)u \quad ; \quad y = x$$

where $b(t) = 1$, $t \in [2n, 2n+1)$, $n \in \mathbf{N}$ and $b(t) = -1$ otherwise. Clearly, although $b(t)$ is discontinuous, it can be approximated within a small error in the mean-square sense resulting in an alternative description for the plant

$$\dot{x} = b_o(t)u + \tilde{b}(t)u \quad ; \quad y = x$$

where b_o is a smooth function and $\tilde{b} = b - b_o$ is small in the mean-square. For this example, the controllability matrix of the nominal plant is $Q_c(t) = b_o(t)$ and must be equal to zero at some time instants t_j since $b_o(t)$ is continuous. This implies loss of uniform controllability at those time instants and consequently loss of strong controllability in an interval around each t_j.

Thus, in our formulation, we admit a quite general class of plants with piecewise Lipschitz continuous parameters, including cases where smoothness and/or strong controllability/observability are lost during short time periods. Moreover, inside each interval (t_j, t_{j+1}) the nominal part of the plant $[A_o, b_o, c_o]$ satisfies Assumptions 3.1–3.3, uniformly in j. Therefore, inside each (t_j, t_{j+1}), the nominal plant admits an I/O operator description of the form (3.4) or (3.5) in a piecewise sense. This enables us to extend any results obtained for smooth parameters to the case of non-smooth ones, by expressing the effects of the perturbation part $[\tilde{A}, \tilde{b}, \tilde{c}]$ and the discontinuities at t_j as a small in-the-mean-square error.

We note, however, that the stability analysis for systems with discontinuous parameters becomes considerably harder since piecewise stability does not, in general, guarantee closed-loop stability unless some additional conditions are imposed on the average frequency of the discontinuity points.[5] To establish and make this statement precise we use the following notation.

$\{t_j\}_1^\infty$, a strictly increasing sequence in $\mathbf{R}_+$ with $t_j \to \infty$ as $j \to \infty$;

J, a subset of the natural numbers $\mathbf{N}$;

$\mathcal{U}_J(t)$, the characteristic function of the set $\bigcup_{j \in J}[t_j, t_{j+1})$, defined as:

$$\mathcal{U}_J(t) = \left\{ \begin{array}{l} 1 \text{ if } t \in [t_j, t_{j+1}) \text{ for some } j \in J \\ 0 \text{ otherwise} \end{array} \right\}$$

n_J, $\bar{n}_J \in \mathbf{N}$, the number of subintervals (t_j, t_{j+1}) of an interval $[t_0, t_0 + T]$ for which $j \in J$ and the number of transitions from an interval for which $j \in J$ to one for which $j \notin J$, respectively.[6] More precisely, for $T, t_0 \geq 0$, let m, $n \in \mathbf{N}$ such that $t_0 \in [t_m, t_{m+1})$ and $t_0 + T \in (t_n, t_{n+1}]$. Then,

$$n_J = \sum_{j=m}^{n} \mathcal{U}_J(t_j) \quad ; \quad \bar{n}_J = [1 - \mathcal{U}_J(t_m)] + \sum_{j=m+1}^{n} \max\{[\mathcal{U}_J(t_{j-1}) - \mathcal{U}_J(t_j)], 0\}$$

3.7 Lemma: *Consider the system $\dot{x} = A(t)x$ where $A(t)$ is a matrix with piecewise continuous, UB elements. Further, assume that there exist two positive constants k, a such that $\forall\, j \in J \subset \mathbf{N}$*

$$||\Phi(t,\tau)|| \leq k e^{-a(t-\tau)} \;, \quad \forall\, t, \tau \in (t_j, t_{j+1});\; t \geq \tau$$

where $\Phi(.,.)$ is the state transition matrix associated with $A(t)$. Then, the system $\dot{x} = A(t)x$ is ES with rate $-\lambda$, for some constant $\lambda > 0$, if there exists a constant $C \geq 0$ such that

$$-\, a \int_{t_0}^{t_0+T} \mathcal{U}_J(t)\, dt + b \int_{t_0}^{t_0+T} [1 - \mathcal{U}_J(t)]\, dt + n_J \ln(k) + \bar{n}_J \ln(k') \leq -\lambda T + C \tag{3.9}$$

[5] This issue is of particular interest in our study where we intend to design controllers for systems with jump parameters in a piecewise sense, i.e., design the controller as to make the closed-loop ES, inside every interval (t_j, t_{j+1}).

[6] That is, consecutive intervals for which $j \notin J$ count as one.

$\forall t_0 \geq 0$, $\forall T \geq 0$. ▽▽

Proof: It suffices to show that, under the conditions stated above, if x satisfies the ODE $\dot{x} = A(t)x$, then for some constant $K > 0$ and for all $T \geq 0$, $t_0 \geq 0$,

$$||x(t_0 + T)|| \leq K e^{-\lambda T} ||x(t_0)|| \tag{3.10}$$

Let $t \in (t_j, t_{j+1})$. If $j \in J$ we have that

$$||x(t)|| \leq k e^{-a(t-t_j)} ||x(t_j)|| \tag{3.11}$$

On the other hand, since $A(t)$ is UB, there exist constants k', b such that

$$||\Phi(t,\tau)|| \leq k' e^{b(t-\tau)} \ ; \quad \forall t \geq \tau$$

Hence, if $j \notin J$,

$$||x(t)|| \leq k' e^{b(t-t_j)} ||x(t_j)|| \tag{3.12}$$

Further, from (3.11), (3.12) and the continuity of $x(t)$ we obtain, grouping together consecutive intervals for which $j \notin J$,

$$||x(t_0 + T)|| \leq k^{n_J} k'^{\bar{n}_J} \exp\left[b \int_{t_0}^{t_0+T} [1 - \mathcal{U}_J(t)]\, dt - a \int_{t_0}^{t_0+T} \mathcal{U}_J(t)\, dt \right] ||x(t_0)||$$

In view of (3.9), the last inequality implies (3.10) with $K = \exp[C]$. □□

Despite its complicated appearance, the condition of Lemma 3.7 is nothing more than an upper bound on the average size (measure) of intervals where $\Phi(.,.)$ may not be exponentially decaying with rate $-a$ and the average number of discontinuities, such that the overall state transition matrix $\Phi(.,.)$ is exponentially decaying with rate $-\lambda$. This situation may arise in a control systems framework when, for example, there is loss of strong controllability/observability of the plant state-space representation inside short time intervals. In such a case, there may not exist a control law which internally stabilizes the plant in those intervals. However, in view of (3.8) and in order to simplify the presentation, we may describe such a situation by an appropriate selection of the modeled and perturbation parts of the plant. That is, without loss of generality, we may select the nominal part of the plant to be piecewise strongly controllable and observable even if the actual plant fails to be so and incorporate the difference in the perturbation part. Consequently, the following simpler version of Lemma 3.7 is adequate for our purposes.

3.8 Corollary: *The result of Lemma 3.7 holds if $J = \mathbf{N}$ and for some constant $\lambda > 0$, there exists a constant $C \geq 0$ such that*

$$n_J \ln(k) \leq (a - \lambda)T + C$$

$\forall t_0 \geq 0$, $\forall T \geq 0$. ▽▽

Again, a condition on the average number of discontinuities is essential in order to guarantee that piecewise ES implies ES (compared with the slowly TV case where pointwise stability implies stability). It is actually quite straightforward to construct counter-examples of piecewise LTI systems which are ES inside every interval but overall unstable, if there is no constraint on the number of discontinuities. A typical and quite illustrative example is given below.

3.9 Example: Consider the system

$$\dot{x} = A(t)x \ ; \quad A(t) = \begin{cases} A_0 & \text{if } t \in [2k, 2k+1) \\ A_0^{\mathrm{T}} & \text{if } t \in [2k+1, 2k+2) \end{cases} , k = 0, 1, \ldots$$

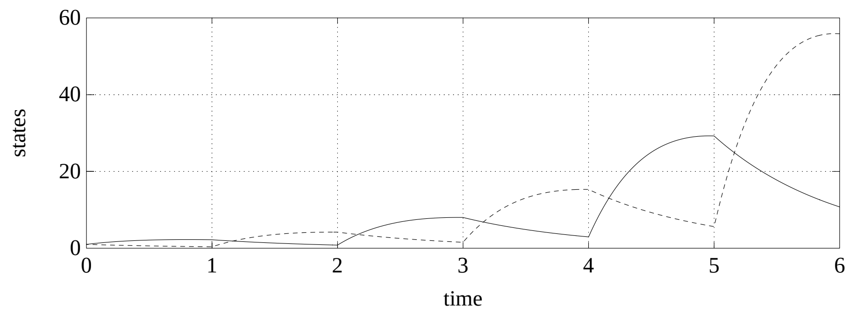

Figure 3.1: Example of instability, occuring when the average number of discontinuities is too large.

where

$$A_0 = \begin{bmatrix} -1 & 5 \\ 0 & -1 \end{bmatrix}$$

Applying Floquet analysis on the above system, we have that

$$x(2k) = \left(e^{A_0^\top} e^{A_0}\right)^k x(0)$$

and the eigenvalues of the matrix $e^{A_0^\top} e^{A_0}$ are easily found to be 0.005 and 3.649. Hence the system is unstable, despite the fact that it is ES inside every interval $(k, k+1)$.

At this point, it is very interesting to perform a simulation of the response of this system. As shown in Fig. 3.1, starting with initial conditions $x(0) = [1, 1]^\top$, inside every interval $(k, k+1)$, one of the states decays as e^{-t+k} while the other decays as e^{-t+k} and $(t-k)e^{-t+k}$. Since, for small positive values of $(t-k)$, the latter is an increasing function of $(t-k)$, the corresponding state increases in magnitude at the beginning of every interval. Thus, if the frequency of discontinuities is too high, the decrease in the magnitude of the states inside each interval may be insufficient to counteract the magnitude increase at the beginning of the interval and instability may occur. $\nabla\nabla$

3.4 Parametric Models of TV I/O Operators

An issue of particular interest in the case of plants with partially known parameters is the design of parameter estimation algorithms, used to identify an unknown I/O operator on-line, from I/O measurements. Such estimators, discussed in more detail in Chapter 6, rely on the ability to describe the unknown operator in an inner product form between a vector of unknown parameters and a vector of signals, often referred to as the regressor vector, which are available for measurement. In this section, our objective is to establish a basic parametrization of certain types of I/O operators having the inner product form that is convenient for parameter estimation.

The following lemma gives a parametric model of a plant in the P_L-form that allows for the identification of the plant I/O operator via parameter estimation.

3.10 Lemma: *Consider a plant described by a strictly proper I/O operator $D_p^{-1}(s,t)N_p(s,t)$, i.e.,*

$$D_p(s,t)[y_p] = N_p(s,t)[u_p] \tag{3.13}$$

where (u_p, y_p) is the I/O pair, $D_p(s,t), N_p(s,t)$ are PDO's in the right form with piecewise continuous, UB coefficients and $D_p(s,t)$ is monic of degree n. Then there exists $\theta_* : \mathbf{R}_+ \mapsto \mathbf{R}^{2n}$ such that, with zero initial conditions,

$$y_p = G(s)[u_p\theta_{1*}] + G(s)[y_p\theta_{2*}] \tag{3.14}$$

where

$$G(s) = q^\top (sI - F)^{-1} \quad ; \quad \theta_*^\top = [\theta_{1*}^\top, \theta_{2*}^\top] \tag{3.15}$$

F is an $n \times n$ Hurwitz matrix and $(q^\top, F)$ is a completely observable pair. $\nabla\nabla$

Proof: Let $D_F(s) = \det(sI - F)$ and operate on (3.13) from the left with $D_F^{-1}(s)$. Then,

$$y_p = D_F^{-1}(s) N_p(s,t)[u_p] + D_F^{-1}(s)\{D_F(s) - D_p(s,t)\}[y_p]$$

Noting that $\deg[D_F(s) - D_p(s,t)] \leq n-1$, a realization of the above equation as in Example 2.37 yields (3.14) and (3.15).

At this point it is worthwhile to perform the state-space analog of this proof. Consider the state-space realization of (3.13) as in Example (2.37):

$$\dot{x} = A(t)x + b(t)u_p \quad ; \quad y_p = c^\top x \tag{3.16}$$

where $A(t)$ contains the coefficients of $D_p(s,t)$ in the left-companion form, $b(t)$ contains the coefficients of $N_p(s,t)$ and $c^\top = [1, 0, \ldots, 0]$. Further, suppose that F is in the left companion form. (There is no loss of generality in such an assumption since $(q^\top, F)$ is a completely observable pair and hence we can always find a (constant) similarity transformation to put it in the left companion form.) We may therefore rewrite (3.16) as

$$\begin{aligned} \dot{x} &= A(t)x + b(t)u_p + \kappa(t)y_p - \kappa(t)c^\top x \\ y_p &= c^\top x \end{aligned}$$

where $\kappa(t)$ is arbitrary. Since $(c^\top, A(t))$ is uniformly observable (Theorem 2.31), there exists $\kappa(t)$ such that $A(t) - \kappa(t)c^\top = F$; in fact it is quite straightforward to construct κ component-wise by taking $\kappa_i(t) = A_{i1}(t) - F_{i1}$, with obvious notation. Using again Example 2.37 and noting that I/O operators are invariant under similarity transformations the proof follows. Finally, it is interesting to observe that a by-product of this analysis is that any initial conditions of the original system are transferred to the modified one, indicating that for arbitrary initial conditions, (3.14) is still valid modulo an exponentially decaying term, depending on the initial conditions. $\square\square$

We should note that the direct analog of this result for plants in the P_R-form is not convenient for estimation purposes since in that case the parametric model contains an internal signal which, in general, is not available for measurement.

A generalization of Lemma 3.10 to systems with the general state-space description (3.8) is given below.

3.11 Lemma: *Consider an LTV plant satisfying Assumptions 3.4–3.6. Then, for any completely observable n-dimensional pair $(q^\top, F)$ there exists a UB, piecewise smooth vector $\theta_* = [\theta_{1*}^\top, \theta_{2*}^\top]^\top$, $\theta_{1*}, \theta_{2*} : \mathbf{R}_+ \mapsto \mathbf{R}^n$ with possible discontinuities at $\{t_j\}_j$ such that the plant is described by the state-space model*

$$\begin{aligned} \dot{x}_F &= Fx_F + \theta_{1*}(t)u_p + \theta_{2*}(t)y_p + \tilde{A}_F(t)x_F + \tilde{b}_F(t)u_p \\ x_F(t_j^+) &= \bar{P}(t_j)x_F(t_j^-) \\ y_p &= q^\top x_F + \tilde{c}_F^\top(t)x_F \end{aligned} \tag{3.17}$$

with the same internal stability properties as the original plant and such that:

- *$\tilde{A}_F, \tilde{b}_F, \tilde{c}_F$ are UB, piecewise continuous matrices for which there exist constants $C, K > 0$ such that*

$$\int_{t_0}^{t_0+T} |\tilde{\cdot}|^2 \leq C + K\mu' T$$

  for all $t_0, T \geq 0$ and μ' as in Assumption 3.4.

- *$\bar{P}$ is a UB piecewise smooth matrix with UB inverse and derivative everywhere except at $t = t_j$, $j = 1, 2, \ldots$. Furthermore, as the size of jumps of the nominal plant parameters and their derivatives approaches zero, $|\bar{P}(t_j) - I|$ and $|\theta_*(t_j^+) - \theta_*(t_j^-)|$ approach zero.*

where t^+, t^- are used to denote right and left limits respectively. It follows that y_p can be expressed as

$$\begin{aligned} y_p(t) &= \{G(s)[u_p\theta_{1*}]\}(t) + \{G(s)[y_p\theta_{2*}]\}(t) \\ &\quad + \{G(s)[\tilde{A}_F x_F + \tilde{b}_F u_p]\}(t) + \tilde{c}_F^\top(t)x_F(t) \\ &\quad + \sum_{t_j \le t} q^\top \Phi_F(t, t_j)[\bar{P}(t_j) - I]x_F(t_j^-) + q^\top \Phi_F(t, t_0)x_F(t_0) \end{aligned}$$

where $G(s) = q^\top(sI - F)^{-1}$ and $\Phi_F(.,.)$ is the state transition matrix associated with F.

Further, for any $\delta > 0$ there exist $\nu_0, \mu_0' > 0$ such that, for any $\nu \in [0, \nu_0)$, $\mu' \in [0, \mu_0')$ and for as long as $(u_p)_t \in L_\infty$, $x_F/m_p^{1/p}$ is UB, where m_p is a normalization signal as in Lemma 2.56 with $u = [u_p, y_p]^\top$.

$\nabla\nabla$

Proof: Since the nominal part of the plant is strongly observable in all intervals (t_j, t_{j+1}), uniformly in j, there exist (Lyapunov) similarity transformations $P_j(t)$ putting the nominal plant in its observable canonical form inside (t_j, t_{j+1}) and such that $P_j(t), P_j^{-1}(t)$ and $\dot{P}_j(t)$, $t \in (t_j, t_{j+1})$ are UB with respect to t and j. Hence, the plant is described by

$$\begin{aligned} \dot{x}_o &= A_o(t)x_o + b_o(t)u_p + \tilde{A}_o(t)x_o + \tilde{b}_o(t)u_p \ ; \quad t \in [t_j, t_{j+1}) \\ y_p &= c_o^\top x + \tilde{c}_o^\top(t)x \end{aligned} \tag{3.18}$$

with boundary conditions arising from the continuity of the state vector of the original state-space description of the plant[7]

$$x_o(t_j) = P_j^{-1}(t_j^+)P_{j-1}(t_j^-)x_o(t_j^-)$$

and $[A_o(t), b_o(t), c_o]$ being in the observable canonical form. Since P_j, P_j^{-1} are UB, Assumption 3.4 implies that the perturbation part in (3.18) also satisfies

$$\int_{t_0}^{t_0+T} |\cdot|^2 \le C + K_o\mu' T$$

for some $C, K_o > 0$, for all $t_0, T \ge 0$.

Further, there exists a possibly discontinuous vector κ_o such that $A_o(t) - \kappa_o(t)c_o$ is the left-companion matrix which is similar to F. Thus, with (q, F) being an n-dimensional completely observable pair, (3.18) can be written in the form of (3.17) after a (constant) similarity transformation, $x_o = P_F x_F$ where

$$\theta_{1*} = P_F^{-1}b_o \ ; \ \theta_{2*} = P_F^{-1}\kappa_o$$

$$\tilde{A}_F = P_F^{-1}[\tilde{A} - \kappa_o\tilde{c}_o^\top]P_F \ ; \ \tilde{b}_F = P_F^{-1}\tilde{b}_o \ ; \ \tilde{c}_F^\top = \tilde{c}_o^\top P_F$$

and

$$\bar{P}(t) = P_F^{-1}P_j^{-1}(t^+)P_{j-1}(t^-)P_F \ ; \quad t \in [t_j, t_{j+1})$$

It is now quite straightforward to integrate (3.17) and obtain the expression for y_p given above. Notice that as the discontinuities in the nominal plant parameters and their derivatives vanish, $P_j(t_j^+)$ approaches $P_{j-1}(t_j^-)$ and hence, the discontinuities in θ_* vanish and $\bar{P}$ approaches the identity matrix.

[7]Notice that the state of the canonical form (3.18) may be discontinuous.

Finally, for the last part of the lemma, consider the state-space description of the plant (3.8). Since (c_o, A_o) is strongly observable inside every (t_j, t_{j+1}), uniformly in j, it follows that for any $\delta' > \delta$ there exists a piecewise smooth, UB κ' such that the STM $\Phi'(.,.)$ of

$$\dot{x} = [A_o(t) - \kappa'(t)c_o^\top(t)]x$$

satisfies

$$|\Phi'(t,\tau)| \le ke^{-\delta'(t-\tau)} \ ; \ \ \forall t \ge \tau \in (t_j, t_{j+1}), \ \forall j$$

where k is a positive constant. For example, such a κ' can be constructed as in the first part of the proof. Hence, choosing $\delta'' \in (\delta, \delta')$, Corollary 3.8 and Assumption 3.6 imply that there exists $\nu_0 > 0$ such that for all $\nu \in [0, \nu_0)$ $\Phi'(.,.)$ is exponentially decaying with rate at most $-\delta''$. Further, rewriting (3.8) as

$$\begin{aligned} \dot{x} &= [A_o(t) - \kappa'(t)c_o^\top(t)]x + [\tilde{A}(t) - \kappa'(t)\tilde{c}^\top(t)]x + [b(t) + \tilde{b}(t)]u_p + \kappa'(t)y_p \\ y_p &= c_o^\top(t)x + \tilde{c}^\top(t)x \end{aligned} \tag{3.19}$$

and invoking Lemma 2.45 and Assumption 3.4 we have that there exists $\mu_0' > 0$ such that for all $\mu' \in [0, \mu_0')$, the system

$$\dot{x} = [A_o(t) - \kappa'(t)c_o^\top(t)]x + [\tilde{A}(t) - \kappa'(t)\tilde{c}^\top(t)]x$$

is ES with rate at most $-\delta$. Since the plant is described by a bounded state-space representation, $(y_p)_t$ is in L_∞ for as long as $(u_p)_t \in L_\infty$. Hence, from Lemma 2.56 and Corollary 2.52 we obtain that $x/m_p^{1/p}$ is UB which, in turn, implies that $x_F/m_p^{1/p}$ is UB since $x_F(t) = P_F^{-1}P_j^{-1}(t^+)x(t)$ inside $[t_j, t_{j+1})$ and $P_j^{-1}(t)$, $t \in (t_j, t_{j+1})$ is UB, uniformly in j. (Also compare with Corollary 2.57). □□

Lemmas 3.10 and 3.11 show that plants satisfying Assumptions 3.4–3.6 or plants with I/O operator $D_p^{-1}(s,t)N_p(s,t)$ can be described by a parametric model of the form

$$y_p(t) = \{G(s)[u_p\theta_{1*}]\}(t) + \{G(s)[y_p\theta_{2*}]\}(t) + \tilde{\eta}(t) \tag{3.20}$$

where $\tilde{\eta}$ is a term appearing in the non-smooth parameter case and whose magnitude depends on the perturbation parameters ν, μ'. The properties of $\tilde{\eta}$ are discussed in some more detail in Chapter 6. Although the plant parametrization (3.20) is not in the inner product form required for the application of standard gradient-based estimators, it can be readily modified invoking the 'Swapping' Lemma 2.59 or Corollary 2.60 to yield

$$\begin{aligned} y_p &= \{G(s)[u_p], G(s)[y_p]\}\theta_* + \eta + \tilde{\eta} \\ &\triangleq w^\top\theta_* + \eta + \tilde{\eta} \end{aligned} \tag{3.21}$$

where, $w^\top = \{G(s)[u_p], G(s)[y_p]\}$ is a vector of signals which can be constructed from purely I/O information. When θ_* is absolutely continuous, an expression for η is directly obtained from Lemma 2.59 as

$$\eta = G(s)\{G'(s)[u_p\Pi_1]\dot{\theta}_*\} + G(s)\{G'(s)[y_p\Pi_2]\dot{\theta}_*\} \tag{3.22}$$

where $G'(s) = (sI - F)^{-1}$. When θ_* contains a jump function, an analogous expression for η, with an additional term describing the effect of the jumps, is obtained from Corollary 2.60.

In a typical parameter estimation problem, the vector θ_* is unknown but u_p, y_p are continuously measured and therefore their filtered values w can be constructed and used to estimate θ_*. Of course, since θ_* is unknown, $\eta, \tilde{\eta}$ are also unknown and cannot be constructed from available measurements. Therefore, $\eta, \tilde{\eta}$ must be treated as 'noise' or modeling error when the parametrization (3.21) is used to estimate θ_* and they must be small in some sense in order for the estimation to be successful. As shown in Chapter 6, θ_* and consequently the unknown I/O operator of the plant in the P_L form can be identified within a small mean-square error in an I/O sense, provided that

- the absolutely continuous part of θ_*, say θ_*^s, is slowly varying in the mean,
- the average number of discontinuities in an interval, ν, is small and
- the parameter μ', characterizing the perturbation part of the plant, is small.

Since in typical applications the plant parameters contain very few discontinuity points inside long time intervals, the most critical parameter of the three is expected to be the speed of variation of θ_*^s. It is therefore desirable to avoid the estimation of any a priori known fast-varying components of θ_*^s, for example, by decomposing the parameter variations into a 'structured' and an 'unstructured' part; such a decomposition of θ_*^s is discussed in the following subsection.

3.4.1 Structured Parameter Variations

Parametric models of the form (3.21) have been widely used for the on-line estimation of the unknown parameters θ_* —resulting in the identification of the unknown plant I/O operator— in the LTI as well as the LTV case [G.S.84, N.A.89, S.B.89, M.G.88, Kre.86, A.J.83]. Naturally, the performance of a parameter estimator based on the linear model (3.21) depends heavily on the size of the 'noise' terms $\eta, \tilde{\eta}$ as well as the speed of variations of the unknown parameters θ_*. It is therefore of interest to exploit any available a priori knowledge about the form or structure of variations of the unknown parameters in order to decrease the size of the effective perturbation and, in particular, the speed of variation of the absolutely continuous part of the unknown parameters. This idea is illustrated in the following example.

3.12 Example: Let us assume that, for the plant of Lemma 3.10, the time variations of $\theta_*(t)$ are of the form

$$\theta_*(t) = \hat{\theta}_* \sin w_0 t \tag{3.23}$$

where the frequency w_0 is constant and known and $\hat{\theta}_*$ is an unknown but constant vector. If we parametrize the plant according to (3.21), we obtain

$$y_p = w^\top \theta_* + \eta$$

where η, given by (3.22), is an unknown signal. It now follows that if we were to use the last equation as a parametric model to estimate θ_*, we should require w_0 to be small in order for the estimation to be successful. Note that in this approach, we effectively treat θ_* as a constant, making no use of the a priori available knowledge about its structure of variation. On the other hand, using the available a priori information on the time-dependence of θ_* in (3.20) we obtain

$$y_p \;=\; G(s)[u_p \hat{\theta}_{1*} \sin w_0 t] + G(s)[y_p \hat{\theta}_{2*} \sin w_0 t]$$

which, by virtue of the Swapping Lemma, becomes

$$\begin{aligned} y_p &= G(s)[u_p \sin w_0 t]\hat{\theta}_{1*} + G(s)[y_p \sin w_0 t]\hat{\theta}_{2*} \\ &= w^\top \hat{\theta}_* \end{aligned} \tag{3.24}$$

where $w^\top = \{G(s)[u_p \sin w_0 t], G(s)[y_p \sin w_0 t]\}$ is a signal which can be constructed from available measurements and $\hat{\theta}_* = [\hat{\theta}_{1*}, \hat{\theta}_{2*}]$ is unknown but constant. Equation (3.24) is now of the same form as the parametrizations obtained in the LTI case without modeling errors or noise. It is therefore possible, using standard estimation techniques, to estimate $\hat{\theta}_*$ within an asymptotically converging to zero error (in an I/O sense). Once the estimate of $\hat{\theta}_*$ is available, we can obtain the estimate of $\theta_*(t)$ from (3.23). Note that, in contrast to the previous case where the parametric model (3.21) was used for estimation, the parametric model (3.24) results in zero estimation error for any value of w_0 (i.e., slow or fast parameter variations). (For details see Chapter 6, Theorem 6.6.) $\nabla\nabla$

Let us now generalize the previous example by assuming that the plant parameters and consequently $\theta_*(t)$ are decomposed as

$$\theta_*(t) = \Pi(t)\hat{\theta}_*(t) \tag{3.25}$$

where $\hat{\theta}_*(t)$ is the unknown, or 'unstructured' part of $\theta_*(t)$ of dimension $l \in \mathbf{N}$ and $\Pi(t)$ is a known matrix with piecewise smooth UB elements. The flexibility of (3.25) in describing fully or partially structured or even unstructured parameter variations is demonstrated by the following simple examples.

1. *'Fully structured variations':* Assume that $\theta_*(t) = A_0 + \sin(wt)A_1$ where w is known and A_0, A_1 are unknown constants. Then,
$$\Pi(t) = [I, \sin(wt)I] \quad ; \quad \hat{\theta}_* = [A_0^\top, A_1^\top]^\top$$

2. *'Partially structured variations':* Assume that $\theta_*(t) = A_0 + \sin[(w+\epsilon)t]A_1 + f(t)A_2$ where w is known, $f(t)$ is an unknown function and ϵ, A_0, A_1, A_2 are unknown constants. Then,
$$\begin{aligned} \Pi(t) &= [I, \sin(wt)I, \cos(wt)I] \\ \hat{\theta}_*(t) &= [A_0^\top + f(t)A_2^\top, \cos(\epsilon t)A_1^\top, \sin(\epsilon t)A_1^\top]^\top \end{aligned}$$

3. *'Unstructured variations':* Assume that $\theta_*(t) = A_0 f(t)$ where both $A_0, f(t)$ are unknown. Then,
$$\Pi(t) = I \quad ; \quad \hat{\theta}_*(t) = \theta_*(t)$$

3.13 Remark: Notice that equation (3.25) can be augmented by an additional term $E^*(t)$ to describe the unknown fast, but small in amplitude, part ('jitter') of the variations of $\theta_*(t)$, i.e.,

$$\theta_*(t) = \Pi(t)\hat{\theta}_*(t) + E^*(t)$$

where $||E^*(t)||$ is small or small in the mean. This description may be useful in applications since, by treating $E^*(t)$ as parameter noise, we can avoid the use of high-dimensional $\hat{\theta}_*(t)$ in modeling the time-variations of $\theta_*(t)$. However, for reasons of clarity and ease of exposition, the discussion of this case is omitted as it presents no additional difficulty in the subsequent analysis. $\nabla\nabla$

By incorporating the knowledge of $\Pi(t)$ in the regressor vector, we may estimate $\hat{\theta}_*(t)$ first and then obtain the estimate of $\theta_*(t)$ as $\theta(t) = \Pi(t)\hat{\theta}(t)$ where $\hat{\theta}(t)$ is the estimate of $\hat{\theta}_*(t)$ at time t. The benefit of this approach becomes clear when we consider the case where the parameters to be estimated, contain fast but known TV elements. Due to the finite speed of adaptation, a general adaptive law is not expected to yield a small prediction error $(w^\top(t)\theta(t) - y(t))$ when θ_* is fast TV. However, if a structured decomposition of the parameters succeeds in achieving a slowly varying $\hat{\theta}_*$, then, estimating the slowly TV component only, it is possible to ensure the smallness of the prediction error, despite the fact that the original parameters may be fast TV.

On the other hand, some caution should be exercised when the structured parameter variations approach is used. One of the shortcomings of this approach is that, in general, the relation between the actual system parameters and the vector θ_* is highly nonlinear, making the derivation of the exact structure of θ_* from the structure of the system parameters difficult. For example, consider an LTV system with I/O operator $D_p^{-1}(s,t)N_p(s,t)$ to be identified. From Lemma 3.10, it follows that the variations of θ_* would have the same structure as the coefficients of $D_p(s,t)$ and $N_p(s,t)$. These coefficients, however, do not necessarily represent physical quantities and, in general, would be nonlinearly related to parameters with physical meaning through a similarity transformation. An additional issue of concern is that although, in principle, the speed of the structured part can be arbitrary, practical considerations put an upper bound on the derivatives of Π for which the results make sense. The reason is that, in general, the sensitivity of the prediction error bounds on the practically unavoidable uncertainties in Π increases as the derivatives of Π become larger. Consequently, if we allow Π to vary arbitrarily fast, we must also require that it is known with a practically unreasonable degree of accuracy. Thus, it should be emphasized that the intended purpose of the structured parameter variations approach is to reduce, rather than eliminate, the effective size of the perturbation introduced by the TV nature of the estimated parameters.

Chapter 4

Model Reference Control

4.1 Introduction

A class of feedback control strategies that has attracted considerable interest in the area of control systems and particularly in adaptive control is Model Reference Control (MRC). The principal idea behind MRC is to design the control law so as to achieve some prescribed tracking performance characteristics. In a typical MRC approach, a reference model is selected describing the desired I/O characteristics of the closed-loop plant, from the reference or command input to the plant output. The control law is then designed so that the I/O operator of the closed-loop plant, from the reference input to the plant output, matches the I/O operator of the reference model.

In the case of LTI plants, the MRC approach effectively amounts to a pole and zero placement design whereby feedback is used to place the plant poles at the desired locations while the plant zeros are cancelled and replaced by the desired zeros. For such a design to be meaningful and feasible, both the plant and the reference model must meet certain conditions. First, from an internal stability point of view, any cancellations must occur in the left half-plane, something that is often referred to as the *minimum phase plant* condition (or assumption). Second, from a realization/implementation point of view, the controller I/O operator should be at least proper, something that translates into a matching condition between the relative degrees of the plant and the reference model.

In this chapter we extend the MRC assumptions and design techniques, which are well understood in the LTI case, to the more intricate case of LTV plants. In particular, we study the design of MRC for LTV plants with the objective of forcing the LTV closed-loop plant to have the same (or approximately the same) I/O operator as a, typically LTI, reference model. We begin with Section 4.2 where we state the control problem for a class of LTV plants with smooth parameters, defined by a set of assumptions which are an extension of the LTI MRC assumptions. In Sections 4.3 and 4.4 we design and analyze MRC schemes meeting the MRC objective. The special case of slowly TV plants is treated in Section 4.5 where we study the applicability of pointwise LTI techniques in the design of (approximate) MRC's. In Section 4.6 we consider a more general class of plants with non-smooth and discontinuous parameters and establish that the MRC design of Sections 4.3 and 4.4 can be extended to this case as well, at the expense of some performance deterioration. We conclude this chapter with Section 4.6 where we present simple examples and simulations illustrating the design and performance of MRC schemes in the LTV case.

4.2 Problem Statement

Consider a SISO LTV plant described by the state-space equations

$$\begin{aligned} \dot{x}_p &= A(t)x_p + b(t)u_p \\ y_p &= c^\top(t)x_p \end{aligned} \tag{4.1}$$

and satisfying Assumptions 3.1–3.3.

The *MRC objective* is defined as follows:

Determine a control input u_p such that the closed-loop plant is internally stable and the plant output y_p tracks the output y_m of the LTI reference model[1]

$$y_m = W_m(s)[r] = k_m D_m^{-1}(s)N_m(s)[r] \tag{4.2}$$

for any UB, piecewise continuous reference input signal r.

In order to design a control law that meets the MRC objective, we need to impose certain additional conditions on the plant and the reference model. For reasons of convenience, these conditions are stated on the PDO factorization of the plant, rather than its state space descripion. Note, however, that such a statement is not restrictive since under Assumptions 3.1–3.2 the plant admits a left or right PDO factorization.

Our first condition concerns the high-frequency gain of the plant, as defined in Section 3.2, which should be bounded away from zero. Inherent in this condition is also the requirement that the relative degree of the plant should be constant.

4.1 Assumption: *The sign of the high-frequency gain, $k_p(t)$, is constant and $k_p(t)$ is smooth, UB and bounded away from zero. Without further loss of generality we assume that, there exists a constant c such that*

$$k_p(t) \geq c > 0$$

$\forall t \geq t_0$. ∎

By virtue of Assumptions 3.1–3.2 and 4.1, the I/O operator of the plant (4.1) admits PDO factorizations in the right form (P_R), i.e.,

$$y_p = k_p(t)N_p(s,t)D_p^{-1}(s,t)[u_p] \tag{4.3}$$

or the left form (P_L), i.e.,

$$y_p = D_p^{-1}(s,t)N_p(s,t)k_p(t)[u_p] \tag{4.4}$$

where $D_p(s,t), N_p(s,t)$ are monic PDO's with UB coefficients and of constant degree, denoted by n, m respectively. Furthermore, in (4.3) $D_p(s,t), N_p(s,t)$ are strongly right coprime while in (4.4) $D_p(s,t), N_p(s,t)$ are strongly left coprime PDO's in $[t_0, \infty)$.

Our second condition on the plant concerns the stability properties of $N_p^{-1}(s,t)$.

4.2 Assumption: *$N_p^{-1}(s,t)$ is an ES PIO with rate bounded from above by $-\alpha$, for some $\alpha > 0$* ∎

In other words, the state transition matrix associated with the differential equation $N_p(s,t)[x] = 0$, say $\Phi_N(.,.)$, is assumed to satisfy $||\Phi_N(t,\tau)|| \leq ke^{-a(t-\tau)}$, for some positive constants k, a and all $t \geq \tau \geq t_0$. The differential equation $N_p(s,t)[x] = 0$ describes what is often referred to as the zero dynamics of the plant. Note that Assumption 4.2 is the LTV generalization of the minimum phase condition in the LTI MRC case.

Further, we assume that the reference model is selected to satisfy:

[1]The selection of the reference model as an LTI one is done for the simplicity of the analysis and design convenience; LTV reference models, satisfying similar assumptions, can be accommodated just as well.

4.3 Assumption: *$D_m(s)$ and $N_m(s)$ are monic and Hurwitz (i.e., their inverses are ES PIO's) with* $\deg[N_m(s)] \leq \deg[D_m(s)] - 1$. ∎

4.4 Assumption: *$W_m(s)$ is designed so that* $\deg[D_m(s)] \leq \deg[D_p(s,t)]$, $\deg[D_m(s)] - \deg[N_m(s)] = \deg[D_p(s,t)] - \deg[N_p(s,t)] (= n - m)$ *and $k_m > 0$, i.e., the LTV plant and the reference model have the same (constant) relative degree $n^* \triangleq n - m$.* ∎

In the subsequent sections and chapters, we refer to Assumptions 4.1–4.4 as the MRC assumptions.

The requirement that the plant output y_p tracks the output of the reference model y_m for any UB reference input r implies that the control input u_p should be chosen such that the closed-loop I/O operator $r \mapsto y_p$ is equal to the I/O operator of the reference model. In the mathematical framework of Chapter 2, this problem can be cast in an elegant way as the solution of a Diophantine equation. The design and I/O properties of such an MRC law are presented in the following section.

4.3 TV MRC Design

Employing the techniques of [DLMS.80] we note that for the P_R plant, (4.3), a stabilizing controller is described by an I/O operator with a left factorization

$$u_p = N_2^{-1}(s,t)N_1(s,t)[y_p] \tag{4.5}$$

for some PDO's $N_1(s,t), N_2(s,t)$. Furthermore, a controller with I/O operator

$$u_p = \frac{k_m}{k_p(t)}N_2^{-1}(s,t)N_1(s,t)[y_p] \tag{4.6}$$

can also be used to stabilize the plant (4.4) in the P_L-form by selecting

$$N_2(s,t) = \frac{1}{k_m}\tilde{N}_2(s,t)N_p(s,t)k_m.$$

With this selection, the PDO $N_p(s,t)k_p(t)$ is directly cancelled by the controller PIO $N_2^{-1}(s,t)$; such a cancellation is permitted since $N_p^{-1}(s,t)$ is assumed to be exponentially stable and $k_p(t)$ is bounded away from zero. Thus, the plant I/O operator becomes effectively $D_p^{-1}(s,t)$, which belongs to the class of plants described by a P_R-form and can therefore be stabilized by a controller with a left factorization. In both cases, we must avoid the appearance of the controller PDO $N_1(s,t)$ as a PDO in the closed-loop I/O operator. This is achieved by writing the controller I/O operator in a proper stable factorization form as

$$u_p = N_2^{-1}(s,t)D(s)D^{-1}(s)N_1(s,t)[y_p]$$

where $D^{-1}(s)$ is ES and $\deg[N_1(s,t)] \leq \deg[D(s)] \leq \deg[N_2(s,t)]$ and then implementing $N_2^{-1}(s,t)D(s)$ in the forward path and $D^{-1}(s)N_1(s,t)$ in the feedback path.

The above discussion motivates the design of a MRC law as given by the following lemma.

4.5 Lemma: *Consider the LTV plant (4.3) or (4.4) and suppose that Assumptions 4.1–4.4 hold. Further, consider the control input defined by*

$$u_p = c_0(t)N_2^{-1}(s,t)D(s)\left[r + D^{-1}(s)N_1(s,t)y_p\right] \tag{4.7}$$

where $D(s)$ is a monic, Hurwitz PDO[2] of degree $n-1$ and such that $N_m(s)$ is a right factor of $D(s)$, i.e., $D(s)$ can be expressed as $D_z(s)N_m(s)$ for some Hurwitz $D_z(s)$; $N_i(s,t)$, $i = 1,2$ are PDO's of degree $n-1$

[2] That is, $D^{-1}(s)$ is an ES PIO.

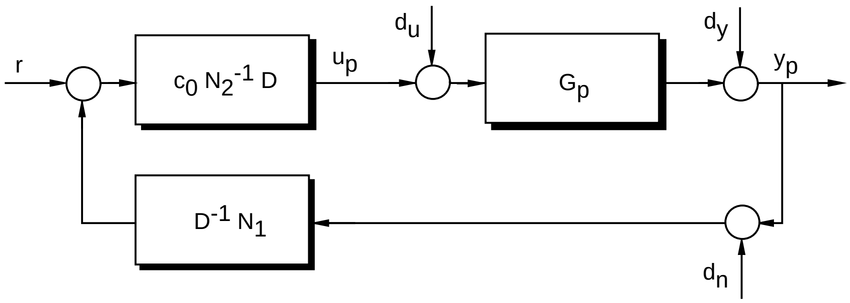

Figure 4.1: The TV MRC closed-loop plant.

with $N_2(s,t)$ monic and $c_0(t)$ is a scalar function of time. Then, the controller parameters, i.e., $c_0(t)$ and the coefficients of $N_i(s,t)$, can be selected so that the closed-loop I/O operator $S_{ry} : r \mapsto y_p$ is BIBO stable and equal to the reference model I/O operator $W_m(s)$; furthermore, the controller parameters are smooth, UB functions of time and can be calculated by solving the algebraic design equations

$$N_2(s,t)c_0^{-1}(t)D_p(s,t) - N_1(s,t)k_p(t)N_p(s,t) = D_z(s)D_m(s)c_0(t)^{-1}N_p(s,t)$$

for a plant with I/O operator in the P_R-form (4.3), or

$$\tilde{N}_2(s,t)D_p(s,t) - k_m N_1(s,t) = k_m D_z(s)D_m(s)k_m^{-1}$$
$$N_2(s,t) = k_m^{-1}\tilde{N}_2(s,t)N_p(s,t)k_m$$

for a plant with I/O operator in the P_L-form (4.4). ▽▽

Proof: In Appendix IV.

Lemma 4.5 establishes the existence and provides a design procedure of a MRC law with smooth UB parameters which achieves the equality of the closed-loop plant and reference model I/O operators. Another important consideration in the design of control systems is the BIBO stability and I/O properties of the closed loop with respect to external inputs entering the loop at any break point between the controller and the plant, as shown in Fig. 4.1. Such external inputs are often used to model the effect of input disturbances (d_u), output disturbances (d_y), sensor noise (d_n) as well as effects of initial conditions. The properties of the I/O operators from any of these external inputs to any closed-loop signal, often referred to as *sensitivity* operators, are discussed in the following lemma.

4.6 Lemma: *Consider the closed-loop system shown in Fig. 4.1 for which $c_0(t)$, $N_1(s,t)$, $N_2(s,t)$ are designed as in Lemma 4.5. Then the I/O dependence of u_p, y_p on the external signals r, d_u, d_y, d_n is described by*

$$\begin{bmatrix} u_p \\ y_p \end{bmatrix} = \begin{bmatrix} S_{ru} & S_{uu} & S_{yu} & S_{nu} \\ S_{ry} & S_{uy} & S_{yy} & S_{ny} \end{bmatrix} \begin{bmatrix} r \\ d_u \\ d_y \\ d_n \end{bmatrix} \tag{4.8}$$

where, omitting the PDO/PIO arguments for simplicity, the various sensitivity operators are given by:
1. *For plants in the P_R-form, (4.3), $S_{ru} = [G_p^R]^{-1}W_m$, $S_{ry} = W_m$ and*

$$\begin{aligned} S_{ru} = D_pD_c^{-1}D \quad &; \quad S_{ry} = k_pN_pD_c^{-1}D \\ S_{uu} = -1 + D_pD_c^{-1}N_2c_0 \quad &; \quad S_{uy} = k_pN_pD_c^{-1}N_2c_0^{-1} \\ S_{yu} = S_{nu} = D_pD_c^{-1}N_1 \quad &; \quad S_{ny} = S_{yy} - 1 = k_pN_pD_c^{-1}N_1 \\ D_c = N_2c_0^{-1}D_p - N_1k_pN_p \quad &= \quad D_zD_mc_0^{-1}N_p \end{aligned} \tag{4.9}$$

2. *For plants in the P_L-form, (4.4), $S_{ru} = [G_p^L]^{-1}W_m$, $S_{ry} = W_m$ and*

$$\begin{aligned} S_{ru} &= k_p^{-1}N_p^{-1}D_pD_c^{-1}k_mD &;\quad& S_{ry} = D_c^{-1}k_mD \\ S_{uu} &= -1 + k_p^{-1}N_p^{-1}D_pD_c^{-1}\tilde{N}_2N_pk_pc_0 &;\quad& S_{uy} = D_c^{-1}\tilde{N}_2N_pk_p \\ S_{yu} = S_{nu} &= k_p^{-1}N_p^{-1}D_pD_c^{-1}k_mN_1 &;\quad& S_{ny} = S_{yy} - 1 = D_c^{-1}k_mN_1 \\ D_c = \tilde{N}_2D_p - k_mN_1 &= k_mD_zD_mk_m^{-1} &;\quad& N_2 = k_m^{-1}\tilde{N}_2N_pk_m \end{aligned} \tag{4.10}$$

Furthermore, there exists $\delta_* > 0$ which in general depends on the stability margin of $D_m(s)$, $D(s)$ and $N_p(s,t)$ such that for any $\delta \in [0, \delta_*)$, and any initial time t_0, the various sensitivity operators are $L_p(\delta)$-stable, $p \in [1, \infty]$, uniformly in t_0; also, for the strictly proper sensitivity operators, the corresponding $g_{p,\delta}$ gains exist and are finite, uniformly in t_0. $\nabla\nabla$

Proof: In Appendix IV.

At this point it must be emphasized that, although the MRC objective is originally motivated as a tracking problem, its practical application should also take into account the effect of possible disturbances and modeling errors on the plant output. As shown in Fig. 4.1, these disturbance effects are modeled by the external inputs d_u, d_y, d_n and their contribution to the total closed-loop response is completely determined by the respective sensitivity operators. These effects can be reduced, for example, by an appropriate selection of the reference model and the auxiliary filter to partially shape the appropriate sensitivity operators, for which $W_m(s)$ and $D(s)$ act as 'tuning' parameters (eqns. (4.9) and (4.10)). Tracking specifications can then be met by prefiltering the reference signal. Some additional sensitivity-shaping techniques are briefly discussed below where we consider the design of higher order TV MRC's which provide some additional degrees of freedom. We refer to such controllers as *over-parametrized TV MRC's*.

4.3.1 Design of Over-Parametrized TV MRC's

A direct consequence of Corollary 2.16 is that the TV MRC solution given by Lemma 4.5 is unique. Its uniqueness, however, relies heavily on our choice to design a TV MRC for which $\deg[N_2(s,t)] = \deg[N_1(s,t)] = n-1$. This choice is by no means necessary and, in fact, a variety of higher order TV MRC's can be produced, all of them satisfying the basic MRC objective as stated above. Despite this obvious disadvantage due to the increased order and complexity, such a TV MRC may have some important advantages in applications, resulting from an additional flexibility of shaping its sensitivity operators to attenuate external disturbances and improve its robustness properties with respect to modeling errors. In the following we briefly discuss this issue by means of two examples.

4.7 Example: *Design of a Strictly Proper TV MRC:*

In this example we consider the design of a TV MRC whose I/O operator $N_2^{-1}(s,t)N_1(s,t)$: $y_p \mapsto u_p$ has relative degree $\ell \geq 1$. Such a design can be obtained as a direct extension of Lemma 4.5 by choosing

- $\deg[N_2(s,t)] = \deg[D(s)] = n + \ell - 1$;
- $\deg[N_1(s,t)] = n - 1$

where, as usual, $n = \deg[D_p(s,t)]$. While it is straightforward to verify that the results of Lemmas 4.5 and 4.6 are still valid —taking of course into account the different degrees of $N_2(s,t)$ and $D(s)$— Corollary 2.16 shows that this TV MRC solution is unique for any fixed $\ell \geq 0$.

Among the advantages of a strictly proper TV MRC is an improved attenuation of high-frequency sensor noise, as it can be seen from the expressions for the sensitivity operator S_{ny} in (4.9) and (4.10). Furthermore, such a design is of particular interest in the adaptive control case, where it contributes to the simplification of the analysis and possibly improves the closed-loop robustness properties. $\nabla\nabla$

4.8 Example: *Shaping the Sensitivity Operators:*

Let us consider an LTV plant in the P_R-form (4.3)[3] for which a TV MRC has been designed as in Lemma 4.5. Following the analysis of [DLMS.80], this basic TV MRC can now be used to derive a class of higher-order controllers which satisfy the same MRC objective and, in addition, allow some flexibility in shaping the characteristics of a closed-loop sensitivity operator.

Omitting the arguments for simplicity, let V, W, D_0, N_0 denote PDO's with UB coefficients such that

- $\deg[V] > \deg[W]$;
- V is a monic PDO and V^{-1} is ES;
- D_0 is monic and $N_0 D_p + D_0 k_p N_p = 0$ in $\mathbf{R}_+$ (the existence of D_0, N_0 is guaranteed by Assumptions 3.1–3.3 which imply the strong right coprimeness of D_p, $k_p N_p$);

Further, consider the control law

$$u_p = c_0 N_x^{-1} V D[r + (VD)^{-1} N_y y_p]$$

where

$$N_x = VN_2 + WN_0 c_0 \quad ; \quad N_y = VN_1 - WD_0$$

where N_2, N_1 and D are as in Lemma 4.5. It is quite straightforward to show that the results of Lemma 4.6 are valid for this control law as well, provided that in the expressions (4.9) we replace N_2, N_1, D, D_z by N_x, N_y, VD, VD_z respectively. Hence, $S_{ry} = W_m(s)$ and since V^{-1} is an ES PIO, this control law also satisfies the MRC objective. It does, however, introduce additional degrees of freedom in the TV MRC solution (namely the arbitrary PDO W) which can be used to alter the properties of the sensitivity operators, other than S_{ry} and S_{ru}.

Suppose, for example, that in addition to the TV MRC objective we would like to reject output disturbances d_y for which an internal model is available. That is, d_y satisfies the differential equation $L(s)[d_y] = 0$; typical examples are constant disturbances ($d_y = const.$, $L(s) = s$) or sinusoids ($d_y = \sin(w_0 t + \varphi)$, $L(s) = s^2 + w_0^2$). For this purpose, we may select W to have degree $\deg[L] - 1$ and satisfy a Diophantine equation of the form

$$XL + WD_0 = VN_1 + VD_z D_m / k_m$$

where X is some PDO of appropriate degree. It follows that this equation has a solution for W and X with UB coefficients provided that D_0 and L are strongly right coprime PDO's. Under this condition, the sensitivity operator S_{yy} in (4.9) takes the form

$$S_{yy} = k_m D_m^{-1} D_z^{-1} V^{-1} XL \, .$$

Thus, the contribution of d_y on the output, given by $S_{yy}[d_y]$, is exponentially decaying to zero.

The above procedure is nothing more than the TV version of the so-called Internal Model Principle (IMP) design which is frequently used to shape the closed-loop sensitivities in the LTI case. Also notice that in this example, the IMP design was facilitated by the controller structure and the assumption that N_p^{-1} is ES which may not be available in a general controller design. (For additional comments, see also the next chapter where an IMP design for P_L plants is considered.) $\nabla\nabla$

In our development and study of the TV MRC problem so far we have dealt with only the I/O operator properties of the closed-loop plant, having tacitly assumed that all initial conditions are equal to zero. To account for arbitrary initial conditions in the closed-loop response we first need to specify the structure of the controller realization and then invoke Lemma 2.35 to establish the internal stability of the closed-loop plant. This problem is discussed in the following section.

[3] Similar techniques can be used for P_L-plants, (4.4), as well.

4.4 Realization of the TV MRC and Internal Stability of the Closed-Loop Plant

The final issues to be resolved in the MRC problem for LTV plants, as posed in the previous section, are the state-space realization of the TV MRC compensators and the internal stability of the closed-loop plant. As mentioned above, the overall TV MRC scheme consists of two compensators, a cascade and a feedback one, with respective I/O operators $c_0(t)N_2^{-1}(s,t)D(s)$ and $D^{-1}(s)N_1(s,t)$. The realization of the two I/O operators in state-space follows the guidelines of the Examples 2.37 and 2.23. Note that the PDO's $N_i(s,t)$, obtained as the solution of the respective Diophantine equations are in the left form and must be converted to the right form before realized in state-space (see Example 2.23). The state-space realization principles of the TV MRC scheme are summarized by the following corollary.

4.9 Corollary: *To realize in state-space the TV MRC scheme of Lemma 4.5 the plant output y_p and input u_p are used to generate a $(2n-1)$-dimensional auxiliary vector ω as follows:*

$$\dot{\omega_1} = F\omega_1 + \theta_1 u_p \;;\; \dot{\omega_2} = F\omega_2 + \theta_2 y_p \;;\; \omega_3 = \theta_3 y_p \tag{4.11}$$

$\omega = [\omega_1^\top, \omega_2^\top, \omega_3]^\top$; $\theta = [\theta_1^\top, \theta_2^\top, \theta_3]^\top$ *is a $(2n-1)$-dimensional parameter vector and $F \in \mathbf{R}^{(n-1)\times(n-1)}$ is a stable matrix with $det(sI-F) = D(s)$. The input to the plant is then taken as*

$$u_p = c_0(t)\left[g^\top \omega + r\right] \tag{4.12}$$

where $g = [q^\top, q^\top, 1]^\top$ is a constant vector such that $(q^\top, F)$ is an observable pair and c_0 is a scalar parameter. Then, there exists a control parameter vector $[\theta_*^\top(t),\, c_{0*}(t)]^\top$ such that the control law (4.11), (4.12) satisfies the TV MRC objective. Further, $[\theta_*^\top(t),\, c_{0*}(t)]^\top$ is UB and at least once differentiable with UB derivative, provided that the plant parameters are UB and possess a sufficiently large but finite number of UB derivatives. ▽▽

Proof: In Appendix IV.

Given the above realization of the controller, we are now in a position to describe the internal stability properties of the closed-loop plant. This result is a direct consequence of Lemmas 2.35 and 4.6 and establishes the well posedness of our solution to the MRC problem. That is, under the MRC assumptions, the TV MRC meets the MRC objective and guarantees the ES stability of the closed-loop plant for all uniform realizations of the plant and its BIBS/BIBO stability with respect to all exogenous signals and initial conditions. This result is made precise by the following theorem.

4.10 Theorem: *Under the conditions given in Lemma 4.5 and Corollary 4.9, the closed-loop plant is ES[4] and, therefore, BIBS stable for any external UB input. Furthermore, there exist constants c, $a > 0$ such that for all $t_0 \geq 0$ and any bounded initial conditions set at t_0, the ZIR of the closed-loop plant is bounded from above by $c\exp[-a(t-t_0)]$; c depends on the bound of the initial conditions and a depends on the location of the roots of $D(s)$, $D_m(s)$ and the rate of exponential stability of $N_p^{-1}(s,t)$.* ▽▽

Proof: In Appendix IV.

Needless to say, the results of the above theorem are also valid for the over-parametrized TV MRC's presented in the previous section, provided that these controllers are realized according to Corollary 4.9; of course, some slight differences appear due to the increased order of the filters F and, for a strictly proper TV MRC, the absence of a direct throughput in the operator $N_2^{-1}(s,t)N_1(s,t) : y_p \mapsto u_p$, i.e., $\theta_3 = 0$.

4.11 Remark: It should be mentioned that a technical, but important, difference between the TV MRC for plants P_L and plants P_R is that the former involves direct cancellation of the plant PDO, while in

[4] Note that an LTV system is ES if the corresponding state transition matrix satisfies $||\Phi(t,\tau)|| \leq ke^{-a(t-\tau)}$, for some positive constants k, a and all $t \geq \tau \geq t_0$.

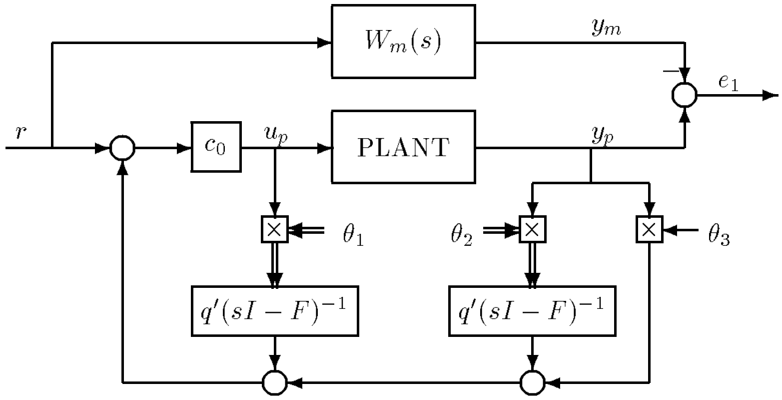

Figure 4.2: The MRC structure for LTV plants (TV MRC).

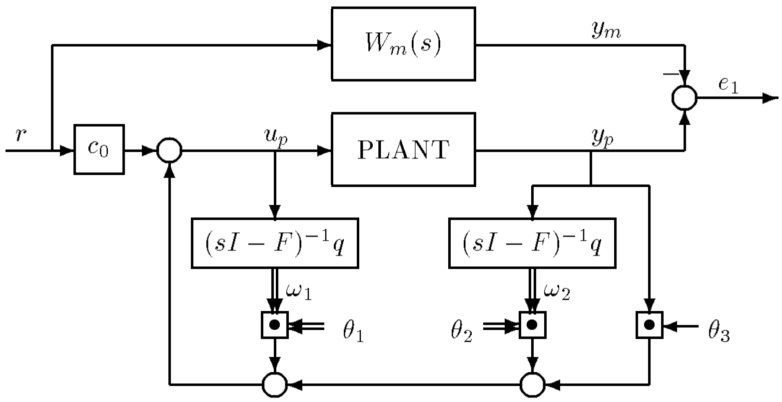

Figure 4.3: The standard (PW) MRC structure for LTI plants.

the latter the plant PDO is cancelled by the closed-loop PIO, after an appropriate solution of the Diophantine equation. The implication of this observation is that the TV MRC state-space structure can be used in both TV MRC or TV PPC design of P_R plants by simply altering the choice of the desired closed-loop PIO. The same is not true for plants P_L for which a TV PPC design requires a different controller structure. $\nabla\nabla$

Following the results of Lemma 4.5 and Corollary 4.9, the block diagram of the closed-loop plant with the TV MRC compensator is as shown in Fig. 4.2. We note, however, that the structure of this TV MRC scheme is essentially different from the standard one, shown in Fig. 4.3, which has been developed and widely used for LTI plants [N.V.78]. The difference between the two controller structures is due to the TV nature of the plant, for which the desired controller parameters are also TV, and it is discussed below.

Using the standard MRC, the control input is generated by

$$\dot{\omega}_1 = F\omega_1 + qu_p \quad ; \quad \dot{\omega}_2 = F\omega_2 + qy_p \tag{4.13}$$

$$u_p = \theta^\top \omega + c_0 r \tag{4.14}$$

where $\omega = [\omega_1^\top, \omega_2^\top, y_p]^\top$; $\theta = [\theta_1^\top, \theta_2^\top, \theta_3]^\top$ and F, q are as in Corollary 4.9. After some straightforward manipulations it follows that u_p can be written as

$$\begin{aligned} u_p &= \bar{M}_2(s,t)D^{-1}(s)[u_p] + \bar{M}_1(s,t)D^{-1}(s)[y_p] + \theta_3(t)y_p + c_0(t)r \\ &= D(s)M_2^{-1}(s,t)\left[c_0(t)r + M_1(s,t)D^{-1}(s)[y_p]\right] \end{aligned} \tag{4.15}$$

where $D(s) = \det(sI - F)$ and $\bar{M}_i(s,t)$ are PDO's such that $\bar{M}_i(s,t)D^{-1}(s) = \theta_i(sI-F)^{-1}q$. From the last equation, it becomes apparent that the cancellation of $D(s)$ and $D^{-1}(s)$ from the I/O operator $y_p \mapsto u_p$ of the controller is not possible in general, unless the PDO's $M_i(s,t)$ are TI. Hence, the matching condition cannot be expressed as a PDO equation and this controller does not satisfy the MRC objective in the general LTV case. Moreover, additional problems arise due to the location of $c_0(t)$, when $N_m(s)$ is not identically one. If, however, we assume that the plant is slowly TV, then we can perform the MRC design in a pointwise fashion that is, as if the plant were LTI at every time instant. We refer to the resulting MRC scheme as the pointwise MRC (PW MRC). The properties of the PW MRC as well as those of the TV MRC in the special but practically interesting case of slowly TV plants are discussed in the following section.

4.5 Slowly TV Plants

Our motivation to consider the case of slowly TV plants lies in the fact that in many practical applications the speed of the plant parameter variations is 'small' in some sense, e.g., the derivatives of the plant parameters are small for all $t \in \mathbf{R}_+$ or they are small in the mean. For reasons of clarity and ease of exposition, we first discuss the case where the plant parameter variations are slow, uniformly in time. The generalization of the results for slowly-in-the-mean TV plants or piecewise smooth plant parameters (e.g., 'jump' parameter variations), requires somewhat more involved arguments and is considered separately later in this chapter.

The intuitive idea behind the analysis of the slowly TV case is to consider a MRC law for the frozen plant (i.e., a pointwise design) for which the resulting frozen closed-loop plant is ES and then use Lemma 2.42 to establish stability for the TV closed-loop. In other words, since ES systems are robust with respect to 'small size' state dependent perturbations, the main difficulty is to establish the appropriate conditions which guarantee that the size of the closed-loop perturbation caused by the variation of the plant parameters is small in some sense. We begin with the analytically simpler case of LTV plants whose parameter variations are slow, uniformly in time. That is, we consider the LTV plant (4.1) which satisfies Assumptions 3.1–3.3 and, therefore, admits an I/O representation of the form (4.3) or (4.4). In addition, we assume that the plant parameters, denoted by the vector Θ_p, satisfy

4.12 Assumption: $\|\frac{d^i}{dt^i}\Theta_p(t)\| \le \mu$, $\forall\, t \in \mathbf{R}_+$, $i = 1, 2, \ldots$ *for some 'small' parameter* $\mu \ge 0$. ∎

In Assumption 4.12 the parameter μ acts as a measure of the speed of variation of the plant parameters, in terms of the maximum magnitude of their derivatives. When $\mu = 0$ the plant parameters are constant and the plant is LTI. When μ is small, the plant parameters change slowly with time and the properties of the LTV plant can be approximated by the properties of the corresponding sequence of frozen LTI plants. For example, for sufficiently small μ, Assumption 3.2 is implied by

4.13 Assumption: *The PW (frozen) controllability and observability matrices of the triple* $[A(t), b(t), c(t)]$ *are strongly nonsingular.* ∎

Note that, given a plant in the form (4.1), Assumption 4.13 is easier to check than its TV counterpart 3.2, since the construction of the PW controllability and observability matrices is identical to the LTI case and does not involve the derivatives of the parameters. Similarly, the strong coprimeness of PDO's can be simply checked by examining the coprimeness of the corresponding families of frozen polynomials (see Section 2.7, Lemma 2.41). The latter can serve to check the validity of Assumption 4.13 by examining the strong PW coprimeness of the polynomials in the frozen I/O representation of the plant, i.e., the numerator and denominator of the PW plant transfer function.

It should be pointed out that Assumption 4.12 is quite strong in the sense that a possibly large — but finite— number of derivatives of the plant parameters are required to be small. Later in this chapter we discuss how this assumption can be replaced by much weaker versions, e.g., $||\dot{\Theta}_p(t)||$ is small almost everywhere or small on the mean-square sense.

Let us now consider the MRC problem for the LTV plant (4.1) satisfying Assumptions 4.12, 4.13, in addition to 3.1, 3.3. Further, regarding the MRC Assumptions, since the plant is slowly TV, we may replace Assumption 4.2 by its pointwise counterpart:

4.14 Assumption: *The roots $\lambda_i(\tau)$ of the polynomial $N_{p\,\tau}(s) \in \{N_{p\,t}(s)\}_t$ satisfy*

$$Re[\lambda_i(t)] \leq -\alpha$$

$\forall t \geq 0$, for some $\alpha > 0$. ∎

Notice that Assumption 4.14 is considerably easier to check than 4.2. The former is simply a condition on the roots of a family of polynomials while the latter requires the calculation of the state transition matrix associated with the differential equation $N_p(s,t)[x] = 0$. Again, for sufficiently small μ, if Assumption 4.14 holds then so does 4.2.

4.15 Corollary: *Consider the LTV plant 4.1 satisfying Assumptions 4.12, 4.13, 3.1, 3.3, 4.1, 4.14, 4.3, 4.4; then there exists $\mu_0 > 0$ such that the results of Lemma 4.5 hold for any $\mu \in [0, \mu_0)$.* ▽▽

Proof: Straightforward from Lemma 4.5, since there exists $\mu_0 > 0$ such that $\forall \mu \in [0, \mu_0)$, Assumptions 4.13, 4.14 imply 3.2, 4.2 (see Lemmas 2.33, 2.41 and 2.42). □□

4.16 Remark: It is possible to relax the strong controllability assumption to a 'strong stabilizability' one, without affecting the tracking performance of the TV MRC for P_L plants, or with an $O(\mu)$ error for P_R plants. This direction, however, is not pursued here as it imposes certain restrictive conditions on the internal structure of the plant (e.g., existence of Lyapunov transformations, structure of uncontrollable modes) in order to assure the internal stability of the closed-loop plant. ▽▽

The stability and tracking properties of the standard PW MRC, designed pointwise in time, can now be derived from those of the TV MRC and are given by the following theorem.

4.17 Theorem: *Consider a slowly TV plant satisfying Assumptions 4.12, 4.13, 3.1, 3.3, 4.1, 4.14, 4.3, 4.4. Then there exists $\mu_1 > 0$ such that the PW MRC (4.13), (4.14) guarantees that the closed-loop plant is ES and, therefore, BIBS stable for any $\mu \in [0, \mu_1)$. Furthermore, the plant output y_p satisfies*

$$y_p = W_m(s)[r] + L_1(s,t)[u_p] + L_2(s,t)[y_p]$$

where $L_1(s,t)$, $L_2(s,t)$ are strictly proper, ES I/O operators with UB parameters and rate that depends on $D^{-1}(s)$, $D_m^{-1}(s)$. In addition, there exists $\delta_* > 0$ such that for any fixed $\delta \in [0, \delta_*)$, $\gamma_{p,\delta}(L_i)$, $g_{p,\delta}(L_i) \leq O(\mu)$ where $p \in [1, \infty]$, $i = 1, 2$ and δ_* depends on the stability margin of $D^{-1}(s), D_m^{-1}(s)$. ▽▽

Proof: In Appendix IV.

The significance of the above theorem is that for slowly TV systems, a controller can be designed and realized in a PW sense. In other words, for the purposes of a control system design, the plant can be assumed to be LTI at every time instant, something that simplifies a great deal all the necessary computations, at the expense of an $O(\mu)$-small deterioration in performance and stability margins (see also [S.A.91] for a more general and quantitative version of this result).

4.6 Non-Smooth Parameter Variations

Let us now consider the design of a MRC for the more general LTV plant

$$\dot{x} \;=\; A_o(t)x + b_o(t)u_p + \tilde{A}(t)x + \tilde{b}(t)u_p$$

$$y_p = c_o^\top(t)x + \tilde{c}^\top(t)x \tag{4.16}$$

whose nominal and perturbation part satisfy Assumptions 3.4–3.6.

By virtue of Assumption 3.5, the nominal part of the plant admits an I/O operator with a PDO factorization as in (4.3) or (4.4), inside every interval (t_j, t_{j+1}). That is, for the nominal part of the plant (4.16)

$$\begin{aligned}\dot{x}_o &= A_o(t)x_o + b_o(t)u_p \\ y_p &= c_o^\top(t)x_o\end{aligned}$$

we may write an I/O operator in the form (4.3) or (4.4), inside every interval (t_j, t_{j+1}). Further, let us assume that, for the nominal part of the plant, the MRC assumptions (4.1–4.4) are satisfied inside every interval, uniformly in j.[5] Then, the controller design procedures developed in Sections 4.3, 4.4 are applicable and we can design a TV MRC in a piecewise sense for each interval (t_j, t_{j+1}). Thus, the control input is determined from (4.7) or its state-space counterpart (4.12) in a piecewise sense.[6] In this section we apply this MRC input to the general LTV plant (4.16) and analyze the closed-loop stability properties.

Note that we should not expect the outcome of such a design procedure to meet the MRC objective exactly. A simple way to demonstrate this fact is to consider a plant with an output vector containing an infinite number of discontinuities. Since the plant output is a discontinuous function of time at an infinite number of points, it cannot be forced to track the continuous output of the reference model, no matter what control input is used (it is assumed, of course, that delta distributions are not admissible as control inputs).

Another issue of concern is that, even if the perturbation matrices $\tilde{A}, \tilde{b}, \tilde{c}$ are identically equal to zero, the piecewise I/O description of the plant is not complete since it does not include the necessary boundary conditions at each discontinuity point. Consequently, a controller which is designed based on such a description is not necessarily a stabilizing one. However, from Theorem 4.10 we have that the TV MRC guarantees the closed-loop exponential stability, whenever the pertinent assumptions are satisfied. In view of Corollary 3.8, a TV MRC could still preserve the closed-loop stability and achieve 'good' tracking in a mean-square sense, provided that on the average the discontinuities are not too frequent and the perturbation part is sufficiently small. The latter can be visualized by considering a plant with parameter discontinuities separated by large time intervals in the time scale of the closed-loop states. In this case, after a parameter discontinuity occurs the closed-loop plant output may depart from its reference trajectory and then converge to it exponentially fast. Thus, if the interval between two successive discontinuities is long enough, the plant output follows the reference trajectory for most of the time. This intuitive idea is made precise in the following theorem where, as in Chapter 3, ν is used to denote the average frequency of parameter discontinuities and μ' to denote the size of possible state perturbations caused, for example, by smooth approximations of non-differentiable parameters.

4.18 Theorem: *Consider the LTV plant (4.16) whose nominal and perturbation parts satisfy Assumptions 3.4–3.6. Further, suppose that the nominal part of the plant $[A_o, b_o, c_o]$ satisfies the MRC Assumptions 4.1–4.4 inside every interval (t_j, t_{j+1}), uniformly in j and the TV MRC control input is designed based on the nominal plant for each interval (t_j, t_{j+1}). Then there exist $\nu_0 > 0$, $\mu_0' > 0$ such that $\forall \nu \in [0, \nu_0)$, $\forall \mu' \in [0, \mu_0')$, the closed-loop plant is ES. Furthermore, there exist positive constants K, K', C such that*

$$\int_{t_0}^{t_0+T} |y_p(t) - y_m(t)|^2\, dt \le C + K\nu T + K'\mu' T$$

[5] For Assumption 4.2 in particular, uniformity in j means that there exist positive constants k, a, independent of j, such that the state transition matrix associated with the differential equation $N_p(s,t)[x] = 0$, say $\Phi_N(.,.)$, satisfies $\|\Phi_N(t,\tau)\| \le ke^{-a(t-\tau)}$, for all $t \ge \tau \in (t_j, t_{j+1})$ and all j.

[6] At the points of discontinuity t_j the control input and the initial conditions of the filters in (4.12) can be arbitrary but UB. For example, a reasonable choice is the respective left limits.

for all $t_0, T \geq 0$. $\nabla\nabla$

Proof: With the results of the previous sections and Corollary 3.8, the proof of the theorem is quite straightforward. Using a uniform realization of the plant and the appropriate controller realization, we obtain a state space representation for the closed-loop plant which is decomposed into a nominal and a perturbation part.

$$\begin{aligned} \dot{x}_c &= A_c(t)x_c + b_c(t)r + \tilde{A}_c(t)x_c + \tilde{b}_c(t)r \\ y_p &= c_c^\top(t)x_c + \tilde{c}_c^\top(t)x_c \end{aligned}$$

where the subscript 'c' denotes the closed-loop states and nominal parameters and '$\tilde{\cdot}$' denotes the closed-loop perturbation part, having the same properties as $\tilde{A}(t), \tilde{b}(t), \tilde{c}(t)$.

From Theorem 4.10 we have that the nominal part is ES inside all intervals (t_j, t_{j+1}) —with rate depending on $D_m^{-1}(s)$, $D^{-1}(s)$, $N_p^{-1}(s,t)$— and therefore, by Corollary 3.8, the nominal closed-loop plant is ES for sufficiently small ν. Hence, invoking Lemma 2.45, the overall closed-loop plant is ES $\forall\,\mu' \in [0, \mu_0')$ and $\forall\,\nu \in [0, \nu_0)$, for some $\mu_0' > 0$, $\nu_0 > 0$. Finally, the expression for $y - y_m$ is obtained by integrating the solutions of the respective differential equations:

$$y_m(t) = c_m(t)\Phi_m(t,t_j)x_m(t_j) + c_m(t)\int_{t_j}^t \Phi_m(t,\tau)b_m(\tau)r(\tau)\,d\tau \tag{4.17}$$

$$\begin{aligned} y_p(t) &= [c_c(t) + \tilde{c}_c(t)]^\top \Phi_c(t,t_j)x_c(t_j) \\ &\quad + [c_c(t) + \tilde{c}_c(t)]\int_{t_j}^t \Phi_c(t,\tau)\left[\tilde{A}_c(\tau)x_c(\tau) + \tilde{b}_c(\tau)r(\tau)\right]\,d\tau \\ &\quad + c_c(t)\int_{t_j}^t \Phi_c(t,\tau)b_c(\tau)r(\tau)\,d\tau + \tilde{c}_c(t)\int_{t_j}^t \Phi_c(t,\tau)b_c(\tau)r(\tau)\,d\tau \end{aligned} \tag{4.18}$$

where $\Phi_c(\cdot,\cdot)$ is the nominal state transition matrix associated with $A_c(t)$ and the subscript 'm' denotes the reference model. The result now follows from the equality of the I/O operators of $r \mapsto y_m$ and that of the nominal part of $r \mapsto y_p$ inside each interval (t_j, t_{j+1}), i.e.,

$$c_m(t)\int_{t_j}^t \Phi_m(t,\tau)b_m(\tau)r(\tau)\,d\tau = c_c(t)\int_{t_j}^t \Phi_c(t,\tau)b_c(\tau)r(\tau)\,d\tau$$

and the boundedness of x_c. Notice that when discontinuities appear in the plant description, the closed-loop I/O operator is equal to the desired one only inside the intervals between discontinuities and y_p may be discontinuous. $\square\square$

An interesting observation is that y_p, given by (4.18), depends now on the complete closed-loop state vector which, in turn, depends on the zero dynamics of the plant. The appearance of the plant zero dynamics on the output has some important consequences in the selection of normalizing signals for the adaptive control case, considered in Chapter 7.

Finally, it should be mentioned that a somewhat more general (and more complicated) version of the above theorem can be established in the case where strong controllability/observability of the nominal plant is lost inside some time intervals of small-in-the-mean length. Such a result follows from Lemma 3.7 using similar arguments as in Theorem 4.18. We omit the details, however, since in our formulation such a situation can be treated by an appropriate selection of the nominal part of the plant. For example, during these intervals we may choose the nominal part as a fixed LTI system satisfying all the pertinent assumptions and incorporate the difference in the perturbation part.

4.6.1 Slowly TV Plants Revisited

In Section 4.5 we established the properties of a PW-designed MRC in the case of slowly TV plants under the quite restrictive Assumption 4.12. With this assumption we required that all the derivatives of the plant parameters, needed in our calculations, should be sufficiently small. It is intuitive, however, that since the frozen closed loop with a PW MRC is ES, the smallness of the first derivative of the plant parameters should suffice to guarantee the closed-loop stability. (Of course, the argument applies as well to the TV MRC structure with parameters designed for the frozen plant.) This observation is quantified in the following corollary.

4.19 Corollary: *Consider a slowly TV plant satisfying the MRC assumptions of Theorem 4.17 except that the plant parameters are only required to be Lipschitz continuous UB functions of time and Assumption 4.12 is replaced by*

4.20 Assumption: $||\dot{\Theta}_p||_\infty \leq \mu$. ∎

Then there exists $\mu_1 > 0$ such that

1. *a PW-designed MRC guarantees that the closed-loop plant is ES and, therefore, BIBS stable for any $\mu \in [0, \mu_1)$;*
2. *the plant output y_p satisfies*

$$y_p = W_m(s)[r] + L_1[x_c] + L_2[r]$$

where L_1, L_2 are strictly proper ES operators and x_c is the state vector of the closed-loop plant;

3. *for any fixed $\mu_1' \in [0, \mu_1)$ there exists $\delta_* > 0$ which depends on the rate of exponential stability of $D^{-1}(s)$, $D_m^{-1}(s)$, $N_p^{-1}(s,t)$ and the value of μ_1' such that for any $\delta \in [0, \delta_*)$, $\gamma_{p,\delta}(L_i)$, $g_{p,\delta}(L_i) \leq O(\mu)$ where $\mu \in [0, \mu_1']$, $i = 1, 2$ and $p \in [1, \infty]$.* ∇∇

Proof: In Appendix IV.

Notice that, as in Theorem 4.18, the complete state vector of the closed-loop plant appears in the output which may now be affected by the zero dynamics of the plant. This is an important qualitative difference from the smooth-parameter case where the stability properties of the output perturbation operators were independent of the rate of exponential stability of $N_p^{-1}(s,t)$ (Theorem 4.17).

4.7 Examples

In the following examples we demonstrate the similarities and differences in the design and tracking performance of the TV and the PW MRC schemes.

In both examples and the corresponding simulations we consider the LTV plant

$$\frac{d^2}{dt^2} y_p + a_1 \frac{d}{dt} y_p + a_2 y_p = u_p \tag{4.19}$$

where a_1, a_2 are TV parameters. The MRC objective is to make the plant output y_p track the output of the LTI reference model

$$[s^2 + 3s + 2] y_m = r \tag{4.20}$$

where r is the reference input signal.

4.21 Example: *PW MRC Design for LTV Plants.* The standard PW MRC law, used for LTI plants, is shown in Fig. 4.3 and is summarized below for the plant (4.19).

$$\dot{\omega}_1 = -\omega_1 + u_p \ , \ \dot{\omega}_2 = -\omega_2 + y_p \ ; \ u_p = \theta_1 \omega_1 + \theta_2 \omega_2 + \theta_3 y_p + r \tag{4.21}$$

where θ_1, θ_2, θ_3 are the scalar controller parameters to be chosen for model-plant I/O matching. Using the properties of the PDO's and PIO's the closed-loop plant may be written as

$$\left[(s+1-\theta_1)(s+1)^{-1}(s^2+a_1s+a_2)(s+1)-\theta_2-\theta_3(s+1)\right](s+1)^{-1}y_p = r \tag{4.22}$$

For I/O matching, i.e., $y_p = y_m$, $\forall r$ we should find θ_1, θ_2, θ_3 such that

$$(s+1-\theta_1)(s+1)^{-1}(s^2+a_1s+a_2)(s+1)-\theta_2-\theta_3(s+1) = (s+1)(s^2+3s+2) \tag{4.23}$$

As shown in [N.V.78],[7] in the special case of LTI plants where a_1, a_2 are constants, there exist constants θ_{*i} for which (4.23) is satisfied. In the TV case, however, a_1, a_2 are functions of time (the argument 't' is dropped for simplicity). Since the PDO's with TV parameters do not commute with respect to multiplication, i.e., $(s+1)^{-1}(s^2+a_1s+a_2)(s+1) \neq s^2+a_1s+a_2$ in general, (4.23) cannot be solved for θ_1, θ_2, θ_3 directly as it is done in the LTI case. Despite this difficulty, expressions for θ_1, θ_2, θ_3 can be obtained by solving (4.23) pointwise in time, i.e., by solving

$$(s+1-\theta_1)\star(s^2+a_1s+a_2)-\theta_2-\theta_3(s+1) = (s+1)(s^2+3s+2) \tag{4.24}$$

for θ_1, θ_2, θ_3, where $P(s,t)\star Q(s,t)$ denotes the pointwise multiplication of two PDO's.[8] The solution of (4.24) is then given as

$$\begin{aligned} \bar\theta_{*1} &= a_1 - 3 \\ \bar\theta_{*2} &= a_1^2 - 4a_1 - a_1a_2 + 3a_2 + 3 \\ \bar\theta_{*3} &= 4a_1 + a_2 - a_1^2 - 5 \end{aligned} \tag{4.25}$$

Using (4.25) in (4.22) the output y_p of the plant is expressed as

$$y_p = (s^2+3s+2)^{-1}r + L(s,t)r \tag{4.26}$$

and the mismatch operator $L(s,t)$ is of the form

$$\begin{aligned} L(s,t) &= -[(s^2+3s+2)(s+1)+X(s,t)]^{-1}X(s,t)(s^2+3s+2)^{-1} \\ X(s,t) &= (a_1-3)[\dot a_1 + (s+1)^{-1}(\dot a_2 - \dot a_1 - \ddot a_1)] \end{aligned} \tag{4.27}$$

In general, the solution (4.25) does not satisfy (4.23), unless $\dot a_1 = \dot a_2 = 0$, $\forall t \geq 0$, i.e., the plant parameters are time-invariant. That is, due to the time variation of the plant parameters, (4.23) can only be solved approximately and the plant I/O operator cannot be made exactly equal to the I/O operator of the reference model with the standard MRC structure. By using an approximate solution of (4.23) (e.g. the pointwise one) we can guarantee stability and small tracking error provided that the plant parameters vary slowly with time. Notice that for slowly TV plants the mismatch operator $L(s,t)$ is stable; this can be seen by writing a, not necessarily minimal, state-space representation of $L(s,t)$ whose a state matrix has diagonal blocks the state matrices corresponding to $(s^2+3s+2)(s+1)$ and $(s+1)$ and at least one of the off-diagonal blocks being $O(\dot a_1, \dot a_2, \ddot a_1)$. $\nabla\nabla$

4.22 Example: *TV MRC Design for LTV Plants.* Using the TV MRC structure, shown in Fig. 4.2, the control law is given as:

$$\dot\omega_1 = -\omega_1 + \theta_1 u_p\ ,\ \dot\omega_2 = -\omega_2 + \theta_2 y_p\ ;\quad u_p = \omega_1 + \omega_2 + \theta_3 y_p + r \tag{4.28}$$

[7] Also follows from Lemma 4.5 and Corollary 4.9 when the plant is LTI.
[8] That is, $s\star a(t) = a(t)\star s = a(t)s$.

Thus, the closed-loop plant can be written as

$$(s+1)^{-1}\left[(s+1-\theta_1)[s^2+a_1s+a_2]-[\theta_2+(s+1)\theta_3]\right]y_p=r \tag{4.29}$$

For model-plant following we should determine θ_1, θ_2, θ_3 such that $y_p=y_m$, $\forall r$, i.e.,

$$(s+1-\theta_1)[s^2+a_1s+a_2]-[\theta_2+(s+1)\theta_3]=(s+1)(s^2+3s+2) \tag{4.30}$$

Comparing (4.30) with (4.23) it is clear that in the former no PIO appears either in the left- or the right-hand side and therefore no commutativity problem arises. From (4.30) we obtain

$$\begin{aligned}(1-\theta_1+a_1-4)s^2+[(1-\theta_1)a_1+\dot{a}_1+a_2-\theta_3-5]s&\\ +(1-\theta_1)a_2+\dot{a}_2-\theta_2-\dot{\theta}_3-2-\theta_3 &= 0\end{aligned}$$

That is, for

$$\begin{aligned}\theta_1=\theta_{*1} &= a_1-3\\ \theta_2=\theta_{*2} &= a_1^2-4a_1-a_1a_2+3a_2+3-5\dot{a}_1+2a_1\dot{a}_1-\ddot{a}_1\\ \theta_3=\theta_{*3} &= 4a_1+a_2-a_1^2-5+\dot{a}_1\end{aligned} \tag{4.31}$$

equation (4.30) is satisfied and the I/O operator of the closed-loop plant is equal to that of the reference model. We note that, in this case, the controller parameters θ_{*1}, θ_{*2}, θ_{*3} are well defined, bounded, smooth functions of time for any bounded, smooth functions a_1, a_2 (i.e., a_1, a_2 may be fast TV).

Further, to demonstrate the results of Theorem 4.17, let us consider again the PW MRC design of the previous example. The control law can then be written as

$$u_p=\bar{\theta}_{*1}(s+1)^{-1}[u_p]+\bar{\theta}_{*2}(s+1)^{-1}[y_p]+\bar{\theta}_{*3}y_p+r$$

or, operating on both sides by $(s+1)$

$$(s+1)[u_p]=\bar{\theta}_{*1}[u_p]+\bar{\theta}_{*2}[y_p]+(s+1)[\bar{\theta}_{*3}y_p]+(s+1)[r]+X_1$$

where $X_1=\dot{\bar{\theta}}_{*1}(s+1)^{-1}[u_p]+\dot{\bar{\theta}}_{*2}(s+1)^{-1}[y_p]$. Furthermore, letting $\tilde{\theta}_i=\bar{\theta}_{*i}-\theta_{*i}$, the PW MRC law becomes

$$(s+1)[u_p]=\theta_{*1}u_p+\theta_{*2}y_p+(s+1)[\theta_{*3}y_p]+(s+1)[r]+X_1+X_2$$

where $X_2=\tilde{\theta}_1u_p+\tilde{\theta}_2y_p+(s+1)[\tilde{\theta}_3y_p]$ and from (4.25) and (4.31), the $\tilde{\theta}_i$'s depend only on the derivatives of the plant parameters. It is now straightforward to verify that

$$y_p=W_m(s)[r]+L_1(s,t)[u_p]+L_2(s,t)[y_p]$$

where

$$L_1(s,t)=W_m(s)(s+1)^{-1}\{\dot{\bar{\theta}}_{*1}(s+1)^{-1}+\tilde{\theta}_1\}$$

$$L_2(s,t)=W_m(s)(s+1)^{-1}\{\dot{\bar{\theta}}_{*2}(s+1)^{-1}+\tilde{\theta}_2\}+W_m(s)\tilde{\theta}_3$$

It is now apparent that the perturbation operators $L_1(s,t), L_2(s,t)$ are strictly proper, ES and their $L_p(\delta)$ gains, $\delta<1$, are $O[\dot{a}_1,\dot{a}_2,\ddot{a}_1]$. $\nabla\nabla$

4.23 Simulations: Let us now simulate the response of the plant (4.19) with the PW and TV MRC for $r=10\sin t$, $a_1=-6$ and $a_2=2\sin\mu t$. The controller parameters are computed using (4.25) for the PW MRC structure (4.21) and (4.31) for the TV MRC structure (4.28). When $\mu=0.1$ the PW MRC results in a bounded but nonzero tracking error while for the TV MRC the tracking error converges to zero (Fig. 4.4).

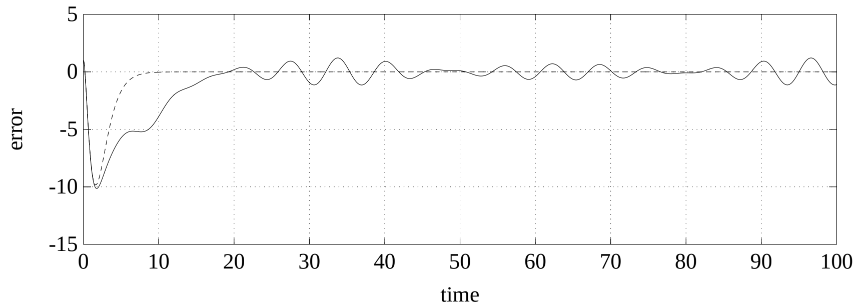

Figure 4.4: MRC tracking error response. Known, slowly TV plant parameters; $\mu = 0.1$. a. (—) PW MRC law; b. (- - -) TV MRC law.

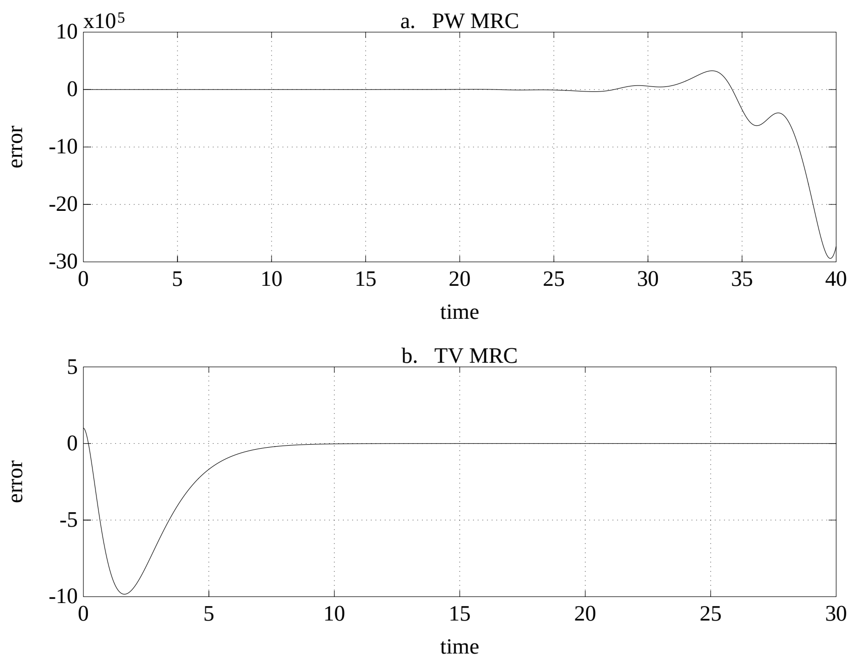

Figure 4.5: MRC tracking error response. Known, fast TV plant parameters; $\mu = 1$. a. PW MRC law: Unbounded response due to fast parameter variations; b. TV MRC law.

Increasing the value of μ to one, however, the tracking error for the PW MRC grows unbounded with time, as shown in Fig. 4.5.a, but the TV MRC still results in a tracking error that converges to zero, as shown in Fig. 4.5.b.

The unbounded closed-loop response with the PW MRC is due to the larger value of μ which results in an unstable mismatch operator $L(s,t)$, given by (4.27). We should note that for this example the solution θ_* for the TV MRC structure is the same as the pointwise solution $\bar{\theta}_*$ for the PW MRC structure, i.e., $\theta_* = \bar{\theta}_*$ when $a_1 = constant$. This demonstrates that the exact model-plant matching achieved by the TV MRC structure is a characteristic of the new structure and not only the specific choice of θ_*. $\nabla\nabla$

APPENDIX IV

Proof of Lemma 4.5:

Plant P_R**:** From (3.4) and (4.7) we obtain that the PIO's involved in the description of the closed-loop system are $D^{-1}(s)$, which is due to the internal cancellations in the control law and is ES by design and $D_c^{-1}(s,t)$, where

$$D_c(s,t) = N_2(s,t)c_0^{-1}(t)D_p(s,t) - N_1(s,t)k_p(t)N_p(s,t)$$

and which should be made ES by an appropriate selection of the controller parameters. Further, after some straightforward calculations, it follows that

$$y_p = k_p(t)N_p(s,t)D_c^{-1}(s,t)D(s)r \triangleq S_{ry}(s,t)r \tag{4.32}$$

where $S_{ry}(s,t) : r \mapsto y_p$ is the closed-loop plant I/O operator. To satisfy the model following objective we need to find $c_0(t)$ and the coefficients of $N_1(s,t)$, $N_2(s,t)$ such that $D_c^{-1}(s,t)$ is ES and $S_{ry}(s,t) = W_m(s)$. Substituting in (4.32) we get

$$N_2(s,t)c_0^{-1}(t)D_p(s,t) - N_1(s,t)k_p(t)N_p(s,t) = D_z(s)D_m(s)k_m^{-1}k_p(t)N_p(s,t) \tag{4.33}$$

which also implies that the closed-loop PIO is ES, since $N_p^{-1}(s,t)$ is ES. For the PDO of the right- and left-hand side of (4.33) to have the same leading coefficient $c_0(t)$ should be selected as

$$c_0(t) = c_{0*}(t) \triangleq k_p^{-1}(t)k_m \tag{4.34}$$

Thus, invoking Corollary 2.16, the Diophantine equation (4.33) can be solved for $N_i(s,t)$ with smooth, UB coefficients. Furthermore, since $N_p^{-1}(s,t)k_p^{-1}(t)$, $D_m^{-1}(s)$, $D_z^{-1}(s)$ and $D(s)$ are all ES PIO's (note that $k_p(t)$ is bounded away from zero), it follows that the controller (4.7), with the so-selected parameters, also guarantees the BIBO stability of $S_{ry}(s,t)$.

Plant P_L**:** From (3.5) and (4.7) we obtain

$$\left\{N_2(s,t)c_0^{-1}(t)k_p^{-1}(t)N_p^{-1}(s,t)D_p(s,t) - N_1(s,t)\right\} y_p = D(s)r \tag{4.35}$$

Letting

$$c_0(t) = c_{0*}(t) \triangleq k_p^{-1}(t)k_m \quad ; \quad N_2(s,t) = k_m^{-1}\tilde{N}_2(s,t)N_p(s,t)k_m \tag{4.36}$$

where $\tilde{N}_2(s,t)$ is a monic PDO of degree $n-m-1$ to be determined, the PIO's involved in the description of the closed loop are $D^{-1}(s)$, $k_p^{-1}(t)N_p^{-1}(s,t)$, due to direct cancellations and $D_c^{-1}(s,t)$ where

$$D_c(s,t) = \tilde{N}_2(s,t)D_p(s,t) - k_mN_1(s,t)$$

Further, the I/O operator $S_{ry}(s,t) : r \mapsto y_p$ is expressed as

$$S_{ry}(s,t) = D_c^{-1}(s,t) k_m D(s) r \tag{4.37}$$

Note that $k_p^{-1}(t) N_p^{-1}(s,t)$ has been cancelled directly by $N_2(s,t)$, an operation that is permitted since $N_p^{-1}(s,t)$ is ES and $k_p(t)$, $k_p^{-1}(t)$ are both UB. Next, in order to meet the control objective $\tilde{N}_2(s,t)$ and $N_1(s,t)$ are selected so that $S_{ry}(s,t) = W_m(s)$, i.e.,

$$\tilde{N}_2(s,t) D_p(s,t) - k_m N_1(s,t) = k_m D_z(s) D_m(s) k_m^{-1} \tag{4.38}$$

As before, the Diophantine equation (4.38) can be solved for $N_1(s,t)$ and $\tilde{N}_2(s,t)$ since both PDO's in the left- and right-hand side of (4.38) are monic and of the same degree $(2n-m-1)$ and the PDO's $D_p(s,t)$ and 1 are always right coprime. Hence, by Corollary 2.16, (4.38), (4.36) can be solved for $N_i(s,t)$ with smooth, UB coefficients. Furthermore, S_{ry} is BIBO stable since all the PIO's involved in its description are ES. □□

Proof of Lemma 4.6:

The part of the lemma regarding the expressions for the sensitivity operators is actually quite straightforward and follows from linearity and the definitions of N_2, N_1 and c_0 as the PDO's satisfying the Diophantine equations given in Lemma 4.5. For the rest, we must first verify that the various expressions make sense as an LTV system description, i.e., that we can write a state space representation for the given I/O operator.

For plants in the P_R-form the operators to be realized are of the form $PD_c^{-1}Q$ where D_c^{-1} is ES and $\deg[D_c] \geq \deg[P] + \deg[Q]$. From example 2.23, the state space realization of the operator $D_c^{-1}Q$ has an input matrix with the top $\deg[D_c] - \deg[Q] - 1$ elements being identically zero (if $\deg[D_c] = \deg[Q]$ then $D_c^{-1}Q$ has a throughput term and $\deg[P] = 0$). Consequently, the output of $D_c^{-1}Q$ can be differentiated at least $\deg[D_c] - \deg[Q] \geq \deg[P]$ times without requiring differentiation of its input. Hence, the output of the operator $PD_c^{-1}Q$ can be obtained as a linear combination of the states of $D_c^{-1}Q$ with weights depending on the coefficients of P and D_c as well as the derivatives of the latter. Furthermore, from the boundedness and smoothness of the coefficients of the various PDO's, it is apparent that the overall STM is ES with the rate of D_c^{-1}.

Similar arguments apply in the case of plants in the P_L-form. In this case, however, we must also realize an operator of the form $D_1^{-1}QD_2^{-1}$ where D_1^{-1}, D_2^{-1} are ES and $\deg[Q] \leq \deg[D_1] + \deg[D_2]$. The easiest way to perform such a realization comes from the fact that the set of PDO's with smooth coefficients is an associative ring (non-commutative, though). Hence, we can apply the Euclidean algorithm to write $Q = Q_1Q_2 + R$ where $\deg[Q_1] \leq \deg[D_1]$, $\deg[Q_2] \leq \deg[D_2]$ and $\deg[R] \leq \deg[Q_1]$. Hence, we can realize $D_1^{-1}QD_2^{-1}$ as a cascade combination of $Q_2D_2^{-1}$ and $D_1^{-1}Q_1$ plus a cascade combination of D_2^{-1} and $D_1^{-1}R$. Again, the overall STM is ES with rate which depends on D_1^{-1} and D_2^{-1}.

It is also interesting to observe that the form of the various PDO's does not affect the result since their parameters are smooth and UB and they can be converted to the appropriate form. That is, denoting by D_R, D_L the right and left form of the same PDO, if the ODE $D_Rx = 0$ is ES then $D_Lx = 0$ is also ES. An alternative way of verifying this is by considering the state space descriptions of D_R^{-1} (controllable canonical form) and D_L^{-1} (observable canonical form). It follows trivially that they are both completely controllable and observable and thus uniform realizations of the same impulse response and topologically equivalent.

Finally, the boundedness and smoothness of the coefficients of the various PDO's together with the ES property of the realizations of the sensitivity operators implies that the associated impulse response $h(t,\tau)$ of the strictly proper part is UB and there exist constants $k, a > 0$ such that $\|h(t,\tau)\| \leq k\exp[-a(t-\tau)]$ for all $t \geq \tau \geq 0$ where a depends on the PIO's in the I/O description of the operator. Thus, $L_p(\delta)$ stability follows from Corollary 2.52 with $\delta_* = a$; the same corollary also shows that for the strictly proper sensitivity operators, the corresponding $g_{p,\delta}$ gains also exist and are finite. □□

Proof of Corollary 4.9:

From (4.11), (4.12) we have

$$u_p = c_0(t)\left\{D^{-1}(s)\bar{N}_2(s,t)[u_p] + D^{-1}(s)\bar{N}_1(s,t)[y_p] + \theta_3(t)y_p + r\right\} \tag{4.39}$$

where

$$\begin{aligned} \bar{N}_2(s,t) &= [Q_1(s),\ldots,Q_{n-1}(s)]\theta_1(t) \\ \bar{N}_1(s,t) &= [Q_1(s),\ldots,Q_{n-1}(s)]\theta_2(t) \end{aligned} \tag{4.40}$$

$D(s) = det(sI-F)$ and $Q_j(s),\ j \in \underline{n-1}$ are TI PDO's such that $q^\top(sI-F)^{-1} = D^{-1}(s)[Q_1(s),\ldots,Q_{n-1}(s)]$. From (4.39) and (4.7) $\bar{N}_i(s,t)$ should satisfy

$$\bar{N}_2(s,t) = [D(s) - N_2(s,t)]c_0^{-1}(t) \quad ; \quad \bar{N}_1(s,t) = N_1(s,t) - D(s)\theta_3(t) \tag{4.41}$$

Note that $\deg[D(s)-N_2(s,t)] = n-2 = \deg[\bar{N}_2(s,t)]$ and that $\deg[N_1(s,t)-D(s)\theta_3(t)] = \deg[\bar{N}_1(s,t)] = n-2$ implies that $\theta_{*3}(t)$ is equal to the leading coefficient of $N_1(s,t)$. Thus, from (4.40) and (4.41), $\theta_i(t)$ should satisfy

$$[Q_1(s),\ldots,Q_{n-1}(s)]\theta_i(t) = \bar{N}_i(s,t)\ ,\ i = 1,2 \tag{4.42}$$

Further, by expressing the PDO's $\bar{N}_i(s,t)$ in the right form, (4.42) yields

$$\theta_{*i}(t) = Q^{-1}\left[\bar{n}_{i1}(t), \bar{n}_{i2}(t), \cdots, \bar{n}_{i\,n-1}(t)\right]^\top \tag{4.43}$$

where $Q = [\underline{q}_1^\top,\ldots,\underline{q}_{n-1}^\top]^\top$, $\underline{q}_j$ are vectors of the coefficients of s^{n-1-j} of $Q_j(s)$ and $\bar{n}_{ij}(t)$ are the coefficients of s^{n-1-j} of $\bar{N}_i(s,t)$ in the right PDO form, $i = 1,2\ ,\ j = 1,\ldots,n-1$, obtained from (4.41). Note that Q^{-1} exists due to the observability of $(q^\top,\ F)$ [Kai.80]. Finally, the boundedness and differentiability properties of $\theta_*(t)$ follow by inspection, directly from (4.43) and the respective Diophantine equation (4.33) or (4.38). □□

Proof of Theorem 4.10:

Observe first that the control law can be put in the form Σ_c of Lemma 2.35 with $u_1 = u = u_p$ and the plant is uniformly realizable. It remains to show that for $r = 0$ and any bounded initial conditions at t_0, $\forall\, t_0 \geq 0$, the signals u_p, y_p decay exponentially fast, uniformly in t_0.

Initial conditions in the auxiliary filters at $t = t_0$ can be introduced directly as a contribution of a term $\Phi_F(t,t_0)w_0$ in the filter states where $\Phi_F(\cdot,\cdot)$ is the STM of F (a matrix exponential if F is TI). It is straightforward to verify that the effect of all such terms can be expressed as an exponentially decaying external input, denoted by w_e, entering the closed-loop system at the reference input node (see Fig. 4.1); consequently, their contribution to u_p, y_p is characterized by the sensitivity operators S_{ru} and S_{ry}. This approach, however, is not used for initial conditions associated with the plant states since the latter is not necessarily ES.

To account for the plant initial conditions we first note that bounded initial conditions of any uniform realization correspond to bounded initial conditions of a canonical form. Further, since the plant is uniformly completely controllable, for any bounded initial conditions $x(t_0)$ there exists $d_c > 0$ and a bounded $\hat{u}(t)$; $t \in [t_0, t_0 + d_c]$ such that [S.A.68]

$$\int_{t_0}^{t_0+d_c} \Phi(t,\tau)B(\tau)\hat{u}(\tau)\,d\tau = \Phi(t_0+d_c,t_0)x(t_0)$$

where $||\hat{u}|| \leq a_1(d_c, ||x(t_0)||)$ and $a_1(\cdot)$ is a constant determined by its arguments. Let us now denote by $x(t)$ the system response with input u_p and initial conditions $x(t_0)$ and by $x_*(t)$ the response with input $u_p + \hat{u}$, for any u_p. Since $\hat{u} = 0$, $\forall\, t > t_0 + d_c$, and employing the linearity assumption, we obtain that $x(t) = x_*(t)$, $\forall\, t > t_0 + d_c$. Also, since the plant is in a bounded realization, $||x(t) - x_*(t)||$ is UB.

In other words, under uniform complete controllability, the fictitious input $\hat{u}$, emulates the effect of arbitrary initial conditions exactly after some finite time, before which all signals are bounded. The net result of the procedure is that the initial conditions of the plant are translated to external inputs which can be manipulated using I/O techniques. These external inputs are $\hat{u}$, entering at the node of input disturbances (d_u in Fig. 4.1), and $\hat{y}$ which is due to the difference between $x(t)$ and $x_*(t)$ and is a bounded, finite function;[9] $\hat{y}$ enters the loop at the node of output disturbances (d_y in Fig. 4.1).

From Lemma 4.6, it follows that the plant input and output satisfy

$$\begin{aligned} u_p &= S_{ru}[\hat{u}] + S_{yu}[\hat{y}] + S_{ru}[w_e] & (4.44)\\ y_p &= S_{yy}[\hat{y}] + W_m[w_e] + S_{uy}[\hat{u}] & (4.45)\end{aligned}$$

where the various sensitivity operators are given in Lemma 4.6 for either P_R or P_L plants. We observe now that w_e is exponentially decaying with the rate of $D^{-1}(s)$ and therefore $w_e \in L_\infty(\delta)$ where $\delta \in [0, \delta_*)$ and δ_* is a positive constant as in Lemma 4.6. Moreover, since $\hat{u}, \hat{y}$ are UB functions of bounded support, $\hat{u}, \hat{y} \in L_\infty(\delta)$. Using Lemma 4.6, the various sensitivity operators are $L_\infty(\delta)$ stable and therefore, $u_p, y_p \in L_\infty(\delta)$ and there exist constants $k, a > 0$, $a < \delta_*$ and k depending on d_c and $x(t_0)$, w_0, such that

$$||[u_p(t), y_p(t)]|| \leq k \exp[-a(t - t_0)]$$

for any initial time $t_0 \geq 0$. Thus, invoking Lemma 2.35 the proof follows. Notice that since ES implies BIBS stability, the closed-loop system is internally stable for any external UB input and any initial conditions, uniformly in t_0. Moreover, this property is shared by any other uniform realization of the plant I/O operator. □□

Proof of Theorem 4.17:

The proof of the theorem relies on the use of swapping techniques to express the PW MRC the control input as

$$\begin{aligned} u_p \;=\;& c_0(t) N_2^{-1}(s,t) N_1(s,t)[y_p] & (4.46)\\ &+ c_0(t) N_2^{-1}(s,t) D(s) \left[r + \hat{L}_1(s,t)[u_p] + \hat{L}_2(s,t)[y_p] \right] \end{aligned}$$

where $N_i(s,t)$ are the PDO's corresponding to the TV MRC and $\hat{L}_i(s,t)$ are proper, stable perturbation operators whose gains are $O(\mu)$. Of course, for the TV MRC design to make sense, μ should be in $[0, \mu_0)$ (see Corollary 4.15). To demonstrate the application of these techniques we derive (4.46) in two ways: one using an I/O approach and property P4 of PDO's and one using a state-space approach and Lemma 2.59.

- *(I/O approach:)* Let $\bar{M}_i(s,t)$ denote the controller PDO's and $\bar{\theta}_3(t)$ the pure gain feedback, corresponding to the standard PW MRC, which are obtained from a pointwise design. Using Property P4 of the PDO's from Chapter 2, the plant input can be written as

$$\begin{aligned} u_p \;=\;& c_0(t) r + D^{-1}(s) D(s) \bar{M}_2(s,t) D^{-1}(s)[u_p]\\ &+ D^{-1}(s) D(s) \bar{M}_1(s,t) D^{-1}(s)[y_p] + \bar{\theta}_3(t) y_p \end{aligned}$$

[9] That is, it is a function of finite (compact) support, vanishing outside a finite interval.

and, consequently,

$$D(s)c_0^{-1}(t)[u_p] = c_0^{-1}(t)\bar{M}_2(s,t)[u_p] + c_0^{-1}(t)\bar{M}_1(s,t)[y_p] + D(s)c_0^{-1}(t)\bar{\theta}_3(t)[y_p] + D(s)[r] + X_2(s,t)D^{-1}(s)[u_p] + X_1(s,t)D^{-1}(s)[y_p] \tag{4.47}$$

where $c_0(t) = k_m/k_p(t)$ and

$$X_i(s,t) = D(s)c_0^{-1}(t)\bar{M}_i(s,t) - c_0^{-1}(t)\bar{M}_i(s,t)D(s)\ ; \quad i = 1,2$$

Thus, using the properties of PDO's, $\deg[X_i(s,t)] \leq 2n-4$ and the coefficients of $X_i(s,t)$, say $\Theta_X(t)$, are of order of the derivatives of $c_0^{-1}(t)$ and $\Theta_M(t)$, the coefficients of $\bar{M}_i(s,t)$. Further, in a PW design $\bar{\theta}_3(t)$ and $\Theta_M(t)$ are obtained from the solution of a PW Diophantine equation which, in turn, can be expressed as $S_0(t)\bar{\Theta}_M(t) = a_0(t)$ where $S_0(t)$ is the PW Sylvester matrix of $D_p(s,t)$ and $k_p(t)N_p(s,t)$, and $S_0(t)$, $a_0(t)$ depend only on the plant parameters $\Theta_p(t)$ and not on their derivatives.[10] Since, under Assumption 4.13, $S_0(t)$ is strongly nonsingular for all t, the PW controller parameters are smooth, UB functions of time and therefore,

$$||\Theta_X(t)|| \leq O(\mu)\,. \tag{4.48}$$

Further, (4.47) can be rewritten as

$$D(s)c_0^{-1}(t)u_p = \bar{N}_2(s,t)u_p + \bar{N}_1(s,t)y_p + D(s)\theta_{*3}(t)y_p + D(s)r + X_2(s,t)D^{-1}(s)u_p + X_1(s,t)D^{-1}(s)y_p + Y_2(s,t)u_p + Y_1(s,t)y_p \tag{4.49}$$

where $\bar{N}_i(s,t), \theta_{*3}(t)$ correspond to the TV MRC design and

$$Y_2(s,t) = c_0^{-1}(t)\bar{M}_2(s,t) - \bar{N}_2(s,t)$$

$$Y_1(s,t) = \underbrace{\left\{c_0^{-1}(t)\bar{M}_1(s,t) - \bar{N}_1(s,t)\right\}}_{Y_{11}(s,t)} + \underbrace{\left\{D(s)[c_0^{-1}(t)\bar{\theta}_3(t) - \theta_{*3}(t)]\right\}}_{Y_{12}(s,t)}$$

After a straightforward comparison of the Diophantine equations for the TV and PW MRC, it follows that $\deg[Y_2(s,t)] \leq n-2$, $\deg[Y_{11}(s,t)] \leq n-2$, $\deg[Y_{12}(s,t)] \leq n-1$ and

$$||\Theta_Y(t)|| \leq O(\mu) \tag{4.50}$$

from which we obtain (4.46) with the input perturbation operators being at least proper, ES with rate depending on $D(s)$ and given by

$$\hat{L}_1(s,t) = D^{-1}(s)X_2(s,t)D^{-1}(s) + D^{-1}(s)Y_2(s,t)$$

$$\hat{L}_2(s,t) = D^{-1}(s)X_1(s,t)D^{-1}(s) + D^{-1}(s)Y_1(s,t)$$

- *(State-Space approach:)* Let $c_0(t)$ and $\bar{\theta}_i(t)$, $i = 1,2,3$, denote the controller parameters of the PW MRC. Then, the control input is given by

$$u_p = G(s)[u_p]\bar{\theta}_1(t) + G(s)[y_p]\bar{\theta}_2(t) + \bar{\theta}_3(t)y_p + c_0(t)r$$

where $G(s)$ denotes the operator

$$G(s):\ u \mapsto q^\top(sI - F^\top)^{-1}[u]$$

[10] Note that in a pointwise approach there is no distinction between the P_R and P_L forms.

and F is the state matrix and q is the input vector of the state-space representation of the auxiliary filters. Letting $G'(s)$ denote the operator

$$G'(s):\ u \mapsto (sI - F^\top)^{-1}[u]$$

and using Lemma 2.59 the term $v_1 = G(s)[u_p]\bar{\theta}_1(t)$ can be written as

$$v_1 = c_0(t)G(s)[u_p\bar{\theta}_1/c_0] + c_0(t)G(s)\left[G'(s)[u_p](\frac{d}{dt}[\bar{\theta}_1/c_0])\right]$$

It now follows from the PW MRC assumptions that $\|\frac{d}{dt}[\bar{\theta}_1/c_0]\|_\infty = O(\mu)$ (similarly for $\bar{\theta}_2$ and y_p). Further, let $\theta_i(t)$ denote the controller parameters of the TV MRC realized with a state matrix $F^\top$ and output vector $q^\top$, i.e., the control law for the TV MRC would be given by

$$u_p' = c_0(t)\left\{G(s)[u_p'\theta_1] + G(s)[y_p'\theta_2] + \theta_3(t)y_p' + r\right\}$$

Next, define $\tilde{\theta}_i = \theta_i - \bar{\theta}_i/c_0$. From the respective Diophantine equations for the PW and TV MRC and Assumption 4.12 it follows that $\|\tilde{\theta}_i\|_\infty = O(\mu)$.

Thus, the PW MRC control input can be expressed as

$$\begin{aligned} u_p &= c_0(t)\{G(s)[u_p\theta_1] + G(s)[y_p\theta_2] + \theta_3(t)y_p + r\} \\ &\quad + \hat{L}_1(s,t)[u_p] + \hat{L}_2(s,t)[y_p] \end{aligned}$$

from which (4.46) follows with the input perturbation operators $\hat{L}_i(s,t)$ being at least proper, ES with rate depending on F and given by

$$\begin{aligned} \hat{L}_1(s,t)[u_p] &= c_0(t)G(s)\left[G'(s)[u_p](\frac{d}{dt}[\bar{\theta}_1/c_0])\right] - c_0(t)G(s)[u_p\tilde{\theta}_1] \\ \hat{L}_2[y_p] &= c_0(t)G(s)\left[G'(s)[y_p](\frac{d}{dt}[\bar{\theta}_2/c_0])\right] \\ &\quad - c_0(t)G(s)[y_p\tilde{\theta}_2] - c_0(t)\tilde{\theta}_3(t)y_p \end{aligned} \tag{4.51}$$

In other words, the closed loop of the LTV plant with the PW MRC can be effectively described as the TV MRC loop perturbed by two dynamic operators $\hat{L}_1$ and $\hat{L}_2$, with respective inputs u_p and y_p and outputs entering the closed-loop system at the node of the reference input (see Fig. 4.1). Furthermore, it is quite straightforward to verify using the results of Chapter 2 that, for any fixed $\delta \in [0, \delta_*)$ (δ_* as in the Theorem) and $p \in [1, \infty]$, $\gamma_{p,\delta}[\hat{L}_i(s,t)]$ and $g_{p,\delta}[\hat{L}_1(s,t)]$ are $O(\mu)$.

Thus, the PW MRC loop without any external inputs is described by

$$\begin{bmatrix} u_p \\ y_p \end{bmatrix} = \begin{bmatrix} S_{ru} \\ S_{ry} \end{bmatrix} v \quad ; \quad v = [\hat{L}_1(s,t), \hat{L}_2(s,t)] \begin{bmatrix} u_p \\ y_p \end{bmatrix}$$

and since $S_{ry} = W_m(s)$ we have that

$$y_p = W_m(s)[r] + L_1(s,t)[u_p] + L_2(s,t)[y_p]$$

where $L_i(s,t) = W_m(s)\hat{L}_i(s,t)$ with the properties stated in the Theorem.[11]

[11] Notice that these expressions are slightly different than the ones obtained in [T.I.87] in that the perturbation operators $L_i(s,t)$ do not depend on the zero-dynamics of the plant; this result is of value in the adaptive control case and its derivation relies heavily on the existence and properties of the TV MRC solution given by Lemma 4.5.

Further, for any $\delta \in [0, \delta_*)$ and $p \in [1, \infty]$ a sufficient condition for the closed-loop $L_p(\delta)$-stability is [D.V.75]

$$\gamma_{p,\delta} \begin{bmatrix} S_{ru} \\ S_{ry} \end{bmatrix} \gamma_{p,\delta}[\hat{L}_1(s,t), \hat{L}_2(s,t)] < 1 \tag{4.52}$$

Since $\gamma_{p,\delta}[\hat{L}_i(s,t)] \leq O(\mu)$ it follows that there exists $\bar{\mu}_1 > 0$ ($\bar{\mu}_1 \leq \mu_0$), depending on the particular choice of p and δ, such that for any $\mu \in [0, \bar{\mu}_1)$ the above inequality is satisfied. Moreover, from Lemma 2.51, if the closed-loop plant (without initial conditions) is $L_p(\delta)$ stable, it is also $L_p(\hat{\delta})$ stable for any $\hat{\delta} \leq \delta$.

To complete the proof we need to show that there exists $\mu_1 > 0$ such that $\forall \mu \in [0, \mu_1)$ the closed-loop system is ES. Let $\delta \in (0, \delta_*)$ and choose, for simplicity, μ_1 such that (4.52) is satisfied for $p = \infty$ and all $\mu \in [0, \mu_1)$. Since the closed loop with the TV MRC is ES, in the presence of arbitrary but bounded initial conditions the truncations of u_p, y_p at t satisfy

$$||[u_p, y_p]_t||_{\infty,\delta} \leq \gamma_{\infty,\delta}(S_r)\gamma_{\infty,\delta}(\hat{L})||[u_p, y_p]_t||_{\infty,\delta} + \beta \sup_{t_0 \leq \tau \leq t} \left[e^{\delta\tau} e^{-a(\tau - t_0)}\right]$$

where β depends on the size of the initial conditions at t_0 (u_p and y_p are taken as 0 for $t < t_0$), $-a \leq -\delta_*$ is the exponential rate of decay of the closed loop with the TV MRC and

$$S_r = \begin{bmatrix} S_{ru} \\ S_{ry} \end{bmatrix} \quad ; \quad \hat{L} = [\hat{L}_1(s,t), \hat{L}_2(s,t)]$$

Hence, for any $\mu \in [0, \mu_1)$,

$$e^{\delta t}||[u_p(t), y_p(t)]|| \leq ||[u_p, y_p]_t||_{\infty,\delta} \leq \frac{\beta e^{\delta t_0}}{1 - \gamma_{\infty,\delta}(S_r)\gamma_{\infty,\delta}(\hat{L})} < \infty; \ \forall t \geq t_0 \geq 0$$

Thus, $||[u_p(t), y_p(t)]|| \leq k e^{-\delta(t - t_0)}$ for some constant k independent of t_0. The proof now follows from Lemma 2.35 and the observations that the PW MRC satisfies the general controller structure requirements of that lemma and under the given assumptions the plant is uniformly completely observable for $\mu < \mu_1$.

It should be pointed out that in an attempt to reduce the conservatism of the result, we should find the supremum of μ_1 with respect to δ and the selection of the —possibly weighted— $L_p(\delta)$-norm. For the former, it is easily obtained that the supremum occurs as $\delta \to 0$ which, of course, implies that increased speeds of parameter variations tend to decrease the exponential stability margin. For the latter, although the selection of p is not straightforward due to the lack of explicit formulas for the $L_p(\delta)$-gains of TV operators, the result of the theorem is still valid with some modifications in the proof. That is, for sufficiently small μ such that (4.52) holds, the closed-loop system is $L_p(\delta)$-stable. Furthermore, for $t > t_0$, the derivatives of u_p, y_p can be expressed as

$$\dot{u}_p = sS_{ru}\left\{\hat{L}_1(s,t)[u_p] + \hat{L}_2(s,t)[y_p]\right\} + \varepsilon$$

$$\dot{y}_p = sS_{ry}\left\{\hat{L}_1(s,t)[u_p] + \hat{L}_2(s,t)[y_p]\right\} + \varepsilon$$

with ε denoting exponentially decaying terms due to the initial conditions. Since $sS_{ru}\hat{L}_2(s,t)$ is the only possibly non-proper term, we substitute the expression for y_p in $\dot{u}_p$ to obtain

$$\dot{u}_p = sS_{ru}\left\{\hat{L}_1(s,t) + \hat{L}_2(s,t)S_{ry}\hat{L}_1(s,t)[u_p] + \hat{L}_2(s,t)S_{ry}\hat{L}_2(s,t)[y_p]\right\} + \varepsilon$$

Thus, with the notation being obvious from the previous relationships,

$$\begin{bmatrix} \dot{u}_p \\ \dot{y}_p \end{bmatrix} = S_L \begin{bmatrix} u_p \\ y_p \end{bmatrix} + \varepsilon$$

where, now, S_L is a proper, $L_p(\delta)$-stable operator. Hence, since $||[u_p, y_p]_t||_{p,\delta}$ is UB, we obtain that $||[\dot{u}_p, \dot{y}_p]_t||_{p,\delta}$ is UB and the proof follows as an application of Lemma 2.55. □□

Proof of Corollary 4.19:

The first part of the theorem is a direct consequence of Lemma 2.42 and the ES property of the frozen closed-loop system with a PW-designed MRC.

To obtain the expression for the plant output, consider a smooth approximation of the plant parameters so that a sufficient number of derivatives are available for the purposes of the TV MRC design. The plant is thus brought in the form of (4.16) with the parameters of the perturbation part being $O[\mu/a]$-small in the uniform norm and a being arbitrary as in Lemma 2.64. From the same lemma, the ith derivative of a parameter approximate is $O[\mu(2a)^{i-1}]$; hence, by Assumptions 4.13 and 4.14 we obtain that for sufficiently small μ, we can select a so that the MRC assumptions are satisfied while $n_I = 1$ in Assumption 3.6 due to the continuity of the plant parameters. At this point and for simplicity we take μ_1 such that the closed loop is ES (from the first part of the proof) and MRC assumptions hold. Hence, we can apply Theorem 4.18 from which the output of the closed-loop plant with the TV MRC designed for the approximate smooth nominal plant is given by (4.18). Thus, for the TV MRC, there are two output perturbation operators of the form

$$L_1'[x_c](t) = [c_c(t) + \tilde{c}_c(t)] \int_{t_0}^{t} \Phi_c(t,\tau)\tilde{A}_c(\tau)x_c(\tau)\, d\tau$$

$$L_2'[r](t) = [c_c(t) + \tilde{c}_c(t)] \int_{t_0}^{t} \Phi_c(t,\tau)\tilde{b}_c(\tau)r(\tau)\, d\tau + \tilde{c}_c(t) \int_{t_0}^{t} \Phi_c(t,\tau)b_c(\tau)r(\tau)\, d\tau$$

which satisfy the properties stated in the corollary. Finally, similar expressions are valid for a PW-designed MRC where the difference between the controller parameters of the TV MRC and its PW counterpart should also be taken into account. As in Theorem 4.17 this difference is $O[\mu]$, for a fixed smooth approximation, and therefore, it can be absorbed in the perturbation part of the closed-loop system leaving the rest of the proof qualitatively unaffected. □□