Chapter 5

TV 'Pole-Placement' Control

5.1 Introduction

One of the main drawbacks of MRC, studied in the previous chapter, is the requirement that the zero dynamics of the plant are ES. Since this requirement may be quite restrictive in applications, it is desirable to develop alternative control strategies —based on a different control objective— which do not suffer the same limitations. In the LTI case, such a strategy is the Pole-Placement Control (PPC) whose objective is to place the poles of the closed-loop system at prescribed locations usually determined by the stability and regulation performance specifications. On the other hand, PPC is rather limited to LTI plants due to the absence of the 'pole' notion in the TV case. We therefore extend the definition of the (TI) PPC objective as to *design a controller such that the PIO's of the closed-loop I/O operator is equal to some prescribed, TV or TI, PIO's.* We refer to this extended objective as the TV PPC objective. For lack of a better name, we refer to a controller that meets the TV PPC objective for an LTV plant, as a TV PPC. When the plant is LTI, the TV PPC objective obviously reduces to the classical PPC objective and is met by a standard PPC scheme. It goes without saying that TI PPC's are a subclass of TV PPC's.

Existing PPC structures for LTV plants (e.g., [M.G.88, Kre.86, G.S.84]), have been derived by pointwise (PW) calculations, based on the 'frozen' plant approach, i.e., under the assumption that the plant is LTI at each time instant. Such controllers, termed as PW PPC, may yield acceptable closed-loop performance, if restricted to the case of slowly TV plants. Of course, a TV PPC is not the same as a PW PPC, the latter being unable to satisfy the control objective or even guarantee stability in the general LTV case.

In this chapter we present the design and analysis of TV PPC schemes for LTV plants. We begin with Sections 5.2, 5.3 where we design and realize in state-space a TV PPC for LTV plants with smooth parameters and establish the closed-loop stability properties. In Section 5.4 we discuss the issue of incorporating some tracking performance features in a TV PPC through the use of internal models. In Section 5.5 we consider the case of non-smooth parameters and generalize the results of Sections 5.2 and 5.3. The special case of slowly TV plants and PW designs is analyzed in Section 5.6. Finally, we present some simple examples and simulations illustrating the design and properties of TV PPC's in Section 5.7.

5.2 TV PPC Design

Consider a SISO LTV plant described by the state-space equations

$$\begin{aligned} \dot{x}_p &= A(t)x_p + b(t)u_p \\ y_p &= c^\top(t)x_p \end{aligned} \tag{5.1}$$

and satisfying Assumptions 3.1–3.3.

As it was pointed out in Chapter 3, Assumptions 3.1–3.2 imply that the I/O operator of the plant (5.1) admits PDO factorizations in the right form (P_R), i.e.,

$$y_p = N_p(s,t)D_p^{-1}(s,t)[u_p] \tag{5.2}$$

or the left form (P_L), i.e.,

$$y_p = D_p^{-1}(s,t)N_p(s,t)[u_p] \tag{5.3}$$

where $D_p(s,t)$ is a monic PDO with UB coefficients and of constant degree, denoted by n and $N(s,t)$ is a PDO of degree $\leq n-1$ with UB coefficients. Furthermore, in (5.2) $D_p(s,t), N_p(s,t)$ are strongly right coprime while in (5.3) $D_p(s,t), N_p(s,t)$ are strongly left coprime PDO's in $[t_0,\infty)$.

The *TV PPC objective* is defined as follows:

Determine a control input u_p such that the closed-loop plant is internally stable and the closed-loop PIO[1] *is equal to a prescribed ES PIO $A_*^{-1}(s,t)$ where $A_*(s,t)$ is a monic PDO of degree $2n-1$ with smooth, UB coefficients.*

In this section we develop and analyze the I/O properties of controllers that meet the TV PPC objective. We start with the following lemma which establishes the existence of a TV PPC and provides the design equations for its construction in an I/O operator form.

5.1 Lemma: *Suppose that for the LTV plant (5.1) Assumptions 3.1–3.3 are satisfied. Then, there exist two $(n-1)$-degree PDO's $N_1(s,t)$, $N_2(s,t)$ with smooth, UB coefficients and $N_2(s,t)$ monic such that the closed-loop PIO of*

a. the P_R plant (5.2) with the controller

$$u_p = -N_2^{-1}(s,t)N_1(s,t)[y_p] \tag{5.4}$$

where $N_1(s,t), N_2(s,t)$ satisfy the Diophantine equation

$$N_2(s,t)D_p(s,t) + N_1(s,t)N_p(s,t) = A_*(s,t)$$

or,

b. the P_L plant (5.3) with the controller

$$u_p = -N_1(s,t)N_2^{-1}(s,t)[y_p] \tag{5.5}$$

where $N_1(s,t), N_2(s,t)$ satisfy the Diophantine equation

$$D_p(s,t)N_2(s,t) + N_p(s,t)N_1(s,t) = A_*(s,t)$$

is equal to the desired PIO $A_*^{-1}(s,t)$. ▽▽

Proof: Straightforward from the expressions for the I/O operators of the plant and the controller and Corollary 2.16. □□

The design of the controller I/O operator as given in the above lemma, with $A_*^{-1}(s,t)$ being ES, also ensures the BIBO stability of the closed-loop system with respect to external inputs. To make this statement more precise, consider the realization of the TV PPC according to Examples 2.38 and 2.39. That is, the control input from the TV PPC is obtained as

$$u_p = -C(s,t)H(s,t)[y_p] \tag{5.6}$$

[1] Modulo, of course, ES PIO's which are due to internal cancellations in the controller.

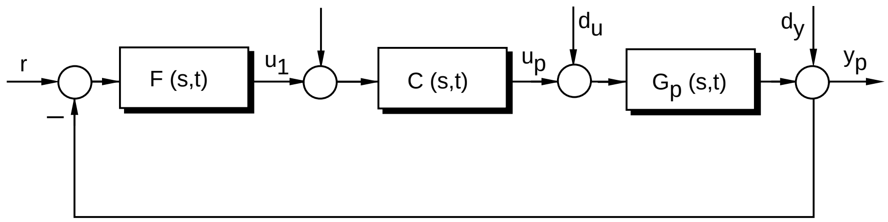

Figure 5.1: The TV PPC closed-loop system.

where, for the P_R plant and the control law (5.4)

$$C(s,t) = N_2^{-1}(s,t)D(s) \quad ; \quad H(s,t) = D^{-1}(s)N_1(s,t)$$

and for the P_L plant and the control law (5.5)

$$C(s,t) = N_1(s,t)D^{-1}(s) \quad ; \quad H(s,t) = D(s)N_2^{-1}(s,t)$$

and $D^{-1}(s)$ is an ES PIO[2] of order $n-1$. This closed-loop configuration is depicted in Fig. 5.1 where the various external signals may represent command inputs (r or v), input disturbances (d_u), output disturbances (d_y), measurement noise (r) or even effects of initial conditions. The total closed-loop response to the various external inputs is simply obtained using superposition, with each input filtered by the appropriate sensitivity operator.

With reference to Fig. 5.1, the following lemma describes the BIBO and $L_p(\delta)$ stability properties of the closed-loop plant with the TV PPC and provides expressions of the various sensitivity operators.

5.2 Lemma: *The I/O description of the closed-loop plant with the TV PPC (5.6), shown in Fig. 5.1, is given by*

$$\begin{bmatrix} u_p \\ y_p \\ u_1 \end{bmatrix} = \begin{bmatrix} S_{ru} & S_{vu} & S_{uu} & S_{yu} \\ S_{ry} & S_{vy} & S_{uy} & S_{yy} \\ S_{r1} & S_{v1} & S_{u1} & S_{y1} \end{bmatrix} \begin{bmatrix} r \\ v \\ d_u \\ d_y \end{bmatrix} \tag{5.7}$$

where, omitting the PDO/PIO arguments for simplicity,

$$S_{yu} = -S_{ru} \quad ; \quad S_{ry} = 1 - S_{yy} \quad ; \quad S_{y1} = -S_{r1}$$

and

1: *for the plant (5.2) in the P_R-form*

$$\begin{aligned} S_{ru} &= D_p A_*^{-1} N_1 \quad ; \quad S_{vu} = D_p A_*^{-1} D \quad ; \quad S_{uu} = 1 - D_p A_*^{-1} N_2 \\ S_{yy} &= 1 - N_p A_*^{-1} N_1 \quad ; \quad S_{vy} = N_p A_*^{-1} D \quad ; \quad S_{uy} = N_p A_*^{-1} N_2 \\ S_{r1} &= D^{-1} N_2 D_p A_*^{-1} N_1 \quad ; \quad S_{v1} = -D^{-1} N_1 N_p A_*^{-1} D \\ & \quad ; \quad S_{u1} = -D^{-1} N_1 N_p A_*^{-1} N_2 \end{aligned} \tag{5.8}$$

2: *for the plant (5.3) in the P_L-form*

$$\begin{aligned} S_{ru} &= N_1 A_*^{-1} D_p \quad ; \quad S_{vu} = N_1 A_*^{-1} D_p N_2 D^{-1} \quad ; \quad S_{uu} = -N_1 A_*^{-1} N_p \\ S_{yy} &= N_2 A_*^{-1} D_p \quad ; \quad S_{vy} = N_2 A_*^{-1} N_p N_1 D^{-1} \quad ; \quad S_{uy} = N_2 A_*^{-1} N_p \\ S_{r1} &= D A_*^{-1} D_p \quad ; \quad S_{v1} = -D A_*^{-1} N_p N_1 D^{-1} \quad ; \quad S_{u1} = -D A_*^{-1} N_p \end{aligned} \tag{5.9}$$

[2] $D(s)$ may be TV with smooth UB coefficients.

Furthermore, there exists $\delta_* > 0$ which in general depends on $A_*(s,t)$ and $D(s)$ such that for any $\delta \in [0, \delta_*)$, and any initial time t_0, the various sensitivity operators are $L_p(\delta)$-stable, $p \in [1, \infty]$, uniformly in t_0; also, for the strictly proper sensitivity operators, the corresponding $g_{p,\delta}$ gains exist and are finite, uniformly in t_0. $\nabla\nabla$

Proof: Straightforward, following similar arguments as in Lemma 4.6. □□

At this point, it should be noted that under Assumptions 3.1–3.3, a plant in the P_L-form can be expressed in the P_R-form and vice-versa. A controller, designed for P_R-plants, has the advantage that it can be implemented so that only fixed PDO's, together with $N_p(s,t)$, appear in the closed-loop I/O operator $v \mapsto y_p$. Such a property is important when the closed-loop response to command signals is considered. Consequently, it is desirable —and feasible— to design the TV PPC for plants in the P_L-form by first converting the plant in the P_R-form and then perform the TV PPC design for the P_R-plant. The apparent drawback of this procedure is an increase in the computational load which may be crucial when the controller calculations are performed on-line. This issue is further discussed in Chapter 8, where an indirect adaptive controller is designed by estimating the plant parameters (necessarily in the P_L form) and calculating the corresponding TV PPC parameters.

Further, overparametrized TV PPC designs can be obtained, as in the TV MRC case, with a higher order controller I/O operator $y_p \mapsto u_p$ (see Examples 4.7 and 4.8). Such a TV PPC may have strictly proper I/O operator and/or possess additional degrees of freedom to allow some partial shaping of the closed-loop sensitivity operators. For example, suppose $-N_1 N_2^{-1}$ is a TV PPC for a P_L plant $D_p^{-1} N_p$, and let V, W, D_0, N_0 denote PDO's with UB coefficients such that

- $\deg[V] > \deg[W]$;
- V is a monic PDO and V^{-1} is ES;
- D_0 is monic and $D_p N_0 + N_p D_0 = 0$ in $\mathbf{R}_+$.

Then the controller $-N_y N_x^{-1}$ with $N_y = N_1 V + D_0 W$ and $N_x = N_2 V + N_0 W$ is also a TV PPC. Taking into account the slight modifications in the degrees of the various PDO's and replacing N_2, N_1, D, A_* in Lemma 5.2 by $N_x, N_y, DV, A_* V$ respectively, the same results are applicable in this case as well. However, notice that the improvement of the properties of the closed loop sensitivity operators via an IMP design may not be as simple as in the TV MRC case, due to the lack of the ES property of N_p^{-1} and, particularly for P_L plants, the different controller factorization.

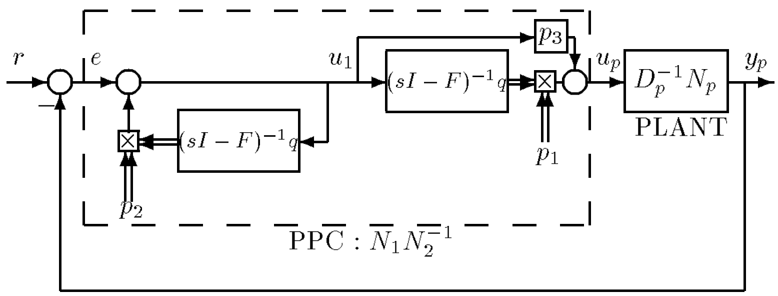

Figure 5.2: The TV PPC structure for LTV plants.

5.3 Realization of the TV PPC and Internal Stability of the Closed-Loop Plant

In order to establish the exponential and, therefore, internal stability of the closed-loop system, we consider a state-space realization of the TV PPC law of Lemma 5.1 according to the guidelines of Examples 2.39 and 2.38. In the case of plants in the P_L-form, the control law is generated by using an auxiliary filter as follows

$$\begin{aligned}\dot{\omega}_1 &= F\omega_1 + qu_1 \\ u_1 &= p_2^\top(t)\omega_1 + (r - y_p) \\ u_p &= p_1^\top(t)\omega_1 + p_3(t)u_1 \end{aligned} \tag{5.10}$$

where r is the reference (command) signal, $u_1 \in \mathbf{R}$ is an internal signal, F is a constant Hurwitz matrix of dimension $(n-1)\times(n-1)$ and (F,q) is a completely controllable pair. The vectors $p_1(t)$, $p_2(t)$ and the scalar $p_3(t)$ are the controller parameters which are to be selected such that the controller has the I/O operator $y_p \mapsto u_p$ specified in Lemma 5.1. The block diagram of the corresponding closed-loop system is shown in Fig. 5.2.

Alternatively, in an I/O operator-notation, the control law (5.10) is expressed as

$$\begin{aligned} u_p &= \{p_1^\top(t)G(s) + p_3(t)\}[u_1] \qquad (5.11) \\ u_1 &= p_2^\top(t)G(s)[u_1] + (r - y_p) \\ G(s) &= (sI - F)^{-1}q \end{aligned}$$

The existence of parameters $p_i(t)$ such that (5.10) or (5.11) represent a TV PPC for a P_L-plant is established in the following corollary.

5.3 Corollary: *Let $N_1(s,t)$, $N_2(s,t)$ be $(n-1)$-degree PDO's with smooth, UB coefficients and with $N_2(s,t)$ monic. Then, there exist smooth, UB parameters $p_i(t)$ such that the I/O operator $y_p \mapsto u_p$ of (5.11) is equal to $-N_1(s,t)N_2^{-1}(s,t)$.* ▽▽

Proof: Immediate from Example 2.38. Notice that the parameters $p_i(t)$ depend on the coefficients of the left form of the PDO's $N_i(s,t)$. These PDO's are initially obtained in the right form as the solution of the corresponding left Diophantine equation (see Lemma 5.1). Consequently, the calculation of the controller parameters $p_i(t)$ involves an additional intermediate step converting right-form PDO's into left-form ones. □□

5.4 Remark: The TV PPC realization for plants in the P_R-form follows similarly by using the results of Example 2.39 to realize the I/O operators $N_2^{-1}(s,t)D(s)$ and $D^{-1}(s)N_1(s,t)$ as in the TV MRC case and is summarized below.[3]

$$\begin{aligned}
\dot{\omega_1} &= F\omega_1 + \theta_1 u_p \\
\dot{\omega_2} &= F\omega_2 + \theta_2 y_p \\
\omega_3 &= \theta_3 y_p \\
u_p &= g^\top \omega + v
\end{aligned} \tag{5.12}$$

where $\omega = [\omega_1^\top, \omega_2^\top, \omega_3]^\top$ is a $(2n-1)$-dimensional vector, $F \in \mathbf{R}^{(n-1)\times(n-1)}$ is a stable matrix with $\det(sI - F) = D(s)$ and $g = [q^\top, q^\top, 1]^\top$ is a constant vector such that $(q^\top, F)$ is an observable pair.

Then, there exists a control parameter vector $[\theta_{1*}^\top, \theta_{2*}^\top, \theta_{3*}]$ such that the I/O operator $y_p \mapsto u_p$ of (5.12) is equal to that of the TV PPC (5.6) as given by Lemma 5.1 for a P_R-plant. (see Corollary 4.9 for details). ▽▽

Having specified the state-space realization of the controller, it is now possible to describe the exponential and BIBS stability properties of the closed-loop plant, as well as the effects of arbitrary initial conditions, for the TV PPC designed in Lemma 5.1 and realized according to Corollary 5.3 or Remark 5.4.

5.5 Theorem: *The closed-loop plant (5.1) with the TV PPC is ES and, therefore, BIBS stable for any external UB input. Furthermore, for arbitrary initial conditions set at t_0, the ZIR of the closed-loop plant decays as $c\exp[-a(t-t_0)]$ where c, a are positive constants independent of t_0; c depends on the size of the initial conditions and a depends on the rate of exponential stability of $A_*^{-1}(s,t)$ and $D^{-1}(s)$.* ▽▽

Proof: The proof is a direct consequence of Lemmas 2.35 and 5.2 and is obtained along the same lines as Theorem 4.10. Notice that for P_L plants, the main difference is that the initial conditions of the filters enter as exponentially decaying disturbances at the 'r' and 'd_u' nodes of Fig. 5.1, while for P_R plants they enter at the 'v' node —same as in the TV MRC case. □□

5.6 Remark: Throughout the development of the TV PPC we have considered a general TV form of the desired PIO $A_*^{-1}(s,t)$ which allows many of the subsequent results to be established in a compact and unified way. In most practical applications, a TI $A_*(s)$ would be sufficient to meet the performance specifications and considerably easier to select and realize. In particular cases, however, it may be possible to exploit the additional flexibility, offered by a TV $A_*(s,t)$, to compensate —at least in part— for time variations in the closed-loop PDO's and improve the closed-loop performance. ▽▽

5.4 Command Tracking with the TV PPC

Although the design of a TV (or PW) PPC scheme is a very general one, including the MRC design as a special case,[4] it does not guarantee any particular tracking performance capabilities for the associated closed-loop system. The reason is that the PPC objective deals mainly with the properties of the closed-loop PIO and state transition matrix, while it is assumed that any tracking performance requirements have already been incorporated in the selection of the desired closed-loop PIO. In the previous chapter we discussed the case where the performance objective was defined in terms of a reference model. This objective, however, required the zero dynamics of the plant to be exponentially stable. It is therefore apparent that, in order to

[3] In this case, it is more convenient to consider v as the command input.

[4] The TV MRC can be obtained from a TV PPC with $A_*(s,t)$ containing $N_p(s,t)$ as a factor and some minor modifications to adjust the high-frequency gain.

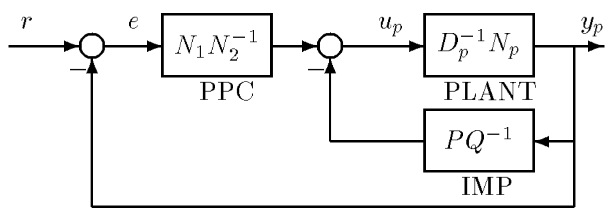

Figure 5.3: TV internal model principle/pole placement controller design.

avoid imposing any such limitations in a PPC design, it is desirable to consider weaker objectives than the MRC when tracking performance requirements should be incorporated in the PPC.

A frequently studied tracking performance objective is to achieve exact tracking for a class of reference inputs whose internal model is specified a priori. In the LTI case the design of a PPC satisfying this objective is a rather straightforward procedure, involving the so-called internal model principle (IMP) [Bng.77]. Briefly described, the essence of this procedure is to introduce certain zeros, corresponding to the internal model, in the sensitivity transfer function from the reference input to the tracking error. The same procedure can be extended to the LTV case, where the condition for the TV IMP/PPC design is a *skew* coprimeness of the internal model and the plant PDO. This condition is essentially similar to that of [Bng.77] in the multivariable LTI case and reduces to the well known coprimeness condition of the internal model and the plant numerator polynomials in the SISO LTI case. In the general LTV case, however, the skew coprimeness cannot be expressed in terms of algebraic equations, as the right or left coprimeness do, which complicates the design of a TV IMP/PPC.

Let us begin our discussion of the TV IMP/PPC design by considering the LTV plant (5.1) and its I/O operator in the P_L form (5.3). The TV IMP/PPC objective is defined as follows.

Determine the control input u_p so as to meet the TV PPC objective and force the output y_p to track reference signals r satisfying the differential equation

$$\Lambda(s)[r] = 0$$

where $\Lambda(s)$ is an a priori specified TI PDO.

In order to achieve exact tracking, the control input u_p is generated as the difference of the output of two compensators as shown Fig. 5.3, i.e.,

$$u_p = N_1(s,t)N_2^{-1}(s,t)[r - y_p] - P(s,t)Q^{-1}(s)[y_p] \tag{5.13}$$

which is realized in state space using ES filters as in the TV PPC case of the previous section.

The first compensator has an input $r - y_p$ and is designed to stabilize the closed loop, while the second has an input y_p and is designed to introduce the internal model of r in the forward path. The properties and assumptions of such a design are summarized in the following corollary.

5.7 Corollary:

a. *Consider the LTV plant (5.1) and let $N_R(s,t), D_R(s,t)$ denote the PDO's of the corresponding P_R form of the plant.*[5] *Also let $Q^{-1}(s)$ be an exponentially stable TI PIO of order $\deg[\Lambda(s)]-1$ and such that $Q(s)$ and $N_R(s,t)$ are strongly left coprime in $\mathbf{R}_+$ and $P(s,t)$ be a PDO of degree $\deg[\Lambda(s)]-1$ with smooth UB coefficients. Further, select $N_2(s,t)$, $N_1(s,t)$ to satisfy the design equations*

$$[D_p(s,t)Q(s)+N_p(s,t)P(s,t)]\tilde{N}_2(s,t)+N_p(s,t)N_1(s,t)=\bar{A}_*(s,t) \qquad (5.14)$$

$$N_2(s,t)=Q(s)\tilde{N}_2(s,t)$$

where $\bar{A}_*^{-1}(s,t)$ is an ES PIO of order $2n-2+\deg[\Lambda(s)]$. Then, the control law (5.13), realized in state space according to Examples 2.39 and 2.38, guarantees the exponential stability of the closed-loop plant. The rate of exponential decay of the closed-loop state transition matrix depends on $\bar{A}_*(s,t)$, $Q(s)$ and the auxiliary filter used in the realization of $N_1(s,t)N_2^{-1}(s,t)$.

b. *Suppose that in (a.) the PDO $P(s,t)$ also satisfies*

5.8 Assumption:

$$X(s,t)\Lambda(s)-N_p(s,t)P(s,t)=D_p(s,t)Q(s) \qquad (5.15)$$

for some PDO $X(s,t)$ with smooth, UB coefficients and degree $n-1$. ∎

Then, in addition to the result in **(a)**, *the tracking error $e=r-y_p$ converges to zero exponentially fast. Such a controller is referred to as TV IMP/PPC.* ∇∇

Proof: In Appendix V.

From the above corollary it is apparent that the main limitation in the design of a TV IMP/PPC lies in the selection of the PDO $P(s,t)$ as to satisfy Assumption 5.8 which is a dynamical equation with respect to the coefficients of the unknown PDO's $P(s,t)$, $X(s,t)$. Since for implementation purposes the controller parameters are required to be UB, we also need to assume the existence of bounded solutions of (5.15). Notice, however, that the exponential stability of the closed-loop plant is guaranteed even if $P(s,t)$ does not satisfy Assumption 5.8, provided of course that it has smooth and UB coefficients.

A general but quite restrictive sufficient condition for (5.15) to have a solution with UB coefficients, for any $D_p(s,t)$, $Q(s)$, is that for any UB smooth function $f(t)$, there exist PDO's $Y_1(s,t)$, $Y_2(s,t)$ with UB coefficients such that

$$Y_1(s,t)\Lambda(s)+N_p(s,t)Y_2(s,t)=f(t) \qquad (5.16)$$

which can be interpreted as skew coprimeness of the PDO's $\Lambda(s)$, $N_p(s,t)$. At this point, it is interesting to notice that considerably weaker conditions can be developed in several special cases of interest, as discussed in the following example.

5.9 Example: Consider the case where $\Lambda(s)=s$ (similar arguments hold for any $\Lambda(s)=s^k$). Then, $P(s,t)=\rho(t)$ and by expressing the PDO's in (5.15) in the left form, it follows that a UB solution for ρ and the coefficients of $X(s,t)$ exists, provided that the ODE

$$N_p(s,t)[\rho]=\alpha(t) \qquad (5.17)$$

has a UB solution, where $\alpha(t)$ is the coefficient of s^0 of the PDO $D_p(s,t)Q(s)$ expressed in the left form. This is trivially true if $\alpha=0$ (in other words, s is a right factor of $D_p(s,t)$) or if $N_p^{-1}(s,t)$ is an ES PIO. These conditions can be further relaxed in the case of periodic systems. For example, if $\alpha(t)$ is a periodic function with period T, then a UB solution of (5.17) can be found, by an appropriate selection of the initial conditions of $\rho(t_0),\dot{\rho}(t_0),\ldots$ provided that the state transition matrix $\Phi_N(\cdot,\cdot)$ of (5.17) satisfies $det[I-\Phi_N(t_0+T,t_0)]\neq 0$. ∇∇

[5] That is, $y_p=N_R(s,t)D_R^{-1}(s,t)[u_p]$.

5.5 Non-Smooth Parameter Variations

In this section we extend the previous design procedures and results to the more general case where the LTV plant is described by

$$\begin{aligned} \dot{x} &= A_o(t)x + b_o(t)u_p + \tilde{A}(t)x + \tilde{b}(t)u_p \\ y_p &= c_o^\top(t)x + \tilde{c}^\top(t)x \end{aligned} \tag{5.18}$$

whose nominal and perturbation part satisfy Assumptions 3.4–3.6.

Due to the discontinuities in the plant parameters, a PDO/PIO factorization may not be available for the LTV plant (5.18) and therefore the TV PPC or TV IMP/PPC design procedures (Lemma 5.1 or Corollary 5.7) are not directly applicable in this case. Under Assumption 3.5, however, the nominal part of the plant possesses a PDO/PIO factorization inside every interval (t_j, t_{j+1}) satisfying Assumptions 3.1–3.3 in a piecewise sense. It is therefore possible to design a TV PPC or TV IMP/PPC[6] for the nominal plant $[A_o, b_o, c_o]$ inside each interval (t_j, t_{j+1}). Thus, the control input is determined from (5.4) or (5.5) or their state-space counterparts (5.12) or (5.10) in a piecewise sense while the parameter discontinuities and the perturbation part of the plant are effectively treated as modeling errors. According to Theorem 5.5, this control input guarantees that the nominal closed-loop is piecewise ES and therefore, invoking Corollary 3.8, ES for a sufficiently small modeling error. We make this idea precise in the following theorem and corollary where we establish the stability and performance properties of such TV PPC designs for the plant (5.18).

5.10 Theorem: *Consider the LTV plant (5.18) whose nominal and perturbation parts satisfy Assumptions 3.4–3.6 with the TV PPC determined from (5.4) or (5.5) in a piecewise sense. Then, there exist $\nu_0 > 0$, $\mu_0' > 0$ such that $\forall \nu \in [0, \nu_0)$, $\forall \mu' \in [0, \mu_0')$, the closed-loop system with the TV PPC (or TV IMP/PPC) is ES with rate depending on $A_*^{-1}(s,t)$ and the values of ν and μ'.* ▽▽

Proof: As in Theorem 4.18.

Notice that, in contrast to the MRC case, the exponential rate of decay of the closed-loop state transition matrix depends now on $A_*(s,t)$ and the values of ν and μ'. Since $A_*(s,t)$ is selected by the designer, the rate of exponential stability of the closed loop can be determined a priori, for sufficiently small ν and μ'.

Finally, for the TV IMP/PPC we can give a characterization of the tracking performance deterioration, due to the parameter discontinuities and the perturbation part of the plant, as stated by the following corollary.

5.11 Corollary: *Under the conditions of Theorem 5.10, suppose that Assumption 5.8 holds inside every interval (t_j, t_{j+1}), uniformly in j and the TV IMP/PPC (5.13) is used to generate the control input. Then, in addition to the results of Theorem 5.10, there exist positive constants K, K', C such that*

$$\int_{t_0}^{t_0+T} |y_p(t) - r(t)|^2 \, dt \leq C + K\nu T + K'\mu' T$$

for all $t_0, T \geq 0$. ▽▽

Proof: As in Theorem 4.18.

5.6 Slowly TV Plants

In the special case where the plant parameters vary slowly with time, the calculation of the controller parameters is considerably simplified by adopting a pointwise approach in the design of a PPC (PW PPC).

[6] Of course, for the TV IMP/PPC, Assumption 5.8 should also be satisfied in a piecewise sense.

This simplification is obtained at the expense of some deterioration in the closed-loop performance defined by the TV PPC objective.

In order to make this idea precise, let us consider the plant (5.1) satisfying Assumptions 3.1, 3.3 and, denoting the plant parameters by the vector Θ_p,

5.12 Assumption: $\|\frac{d^i}{dt^i}\Theta_p(t)\| \leq \mu$, $\forall\, t \in \mathbf{R}_+$, $i = 1, 2, \ldots$ *for some 'small' parameter* $\mu \geq 0$. ∎

5.13 Assumption: *The PW (frozen) controllability and observability matrices of the triple* $[A(t), b(t), c(t)]$ *are strongly nonsingular.* ∎

Under Assumption 5.12, the properties of the LTV plant can be approximated by the properties of the corresponding sequence of frozen LTI plants. For example, for sufficiently small μ, Assumption 3.2 is implied by the easier to check Assumption 5.13. In addition, the validity of Assumption 5.13 may also be checked by examining the strong PW coprimeness of the polynomials in the frozen I/O representation of the plant, i.e., the numerator and denominator of the PW plant transfer function.

Therefore, for sufficiently small μ, Assumptions 3.1, 3.3, 5.12, 5.13, imply that the LTV plant (5.1) admits an I/O representation of the form (5.2) or (5.3), i.e.,

$$y_p = N_p(s,t)D_p^{-1}(s,t)[u_p]$$

or

$$y_p = D_p^{-1}(s,t)N_p(s,t)[u_p]$$

where the coefficients of the PDO's $D_p(s,t), N_p(s,t)$ are slowly TV.[7] Furthermore, for sufficiently small μ, the PDO's $D_p(s,t), N_p(s,t)$ are strongly pointwise coprime and strongly left or right coprime. Consequently, considering the P_L-form of the plant, we may design a PW PPC by taking the controller I/O operator $y_p \mapsto u_p$ to be $-N_1(s,t)N_2^{-1}(s,t)$, i.e.,

$$u_p = -N_1(s,t)N_2^{-1}(s,t)[y_p] \tag{5.19}$$

with $N_1(s,t), N_2(s,t)$ such that

$$D_p(s,t) \star N_2(s,t) + N_p(s,t) \star N_1(s,t) = A_*(s)$$

where '$\star$' denotes pointwise multiplication and $A_*^{-1}(s)$ is the desired, ES PIO[8] of order $2n-1$. The closed-loop stability properties with such a PW PPC are given by the following theorem (similarly for a PW PPC corresponding to the P_R-form of the plant).

5.14 Theorem: *Consider the LTV plant (5.1) satisfying Assumptions 3.1, 3.3, 5.12, 5.13 and its I/O operator expressed in the P_L-form (5.3). Further, consider the PW PPC (5.19), realized in state-space by (5.10) and applied to the LTV plant (5.1). Then there exists a constant $\mu_2 > 0$ such that $\forall\, \mu \in [0, \mu_2)$,*

1. *the closed-loop plant is ES and therefore, BIBS and BIBO stable;*
2. *the closed-loop state transition matrix is exponentially decaying with rate that depends on $A_*^{-1}(s)$, $D^{-1}(s)$ and the value of μ;*
3. *the closed-loop PIO $D_c^{-1}(s,t)$ satisfies*

$$D_c(s,t) = A_*(s) + \Delta(s,t)$$

[7] Note that, in general, the PDO's D_p, N_p of the P_R and P_L form are not the same but their coefficients may be different by $O(\mu)$.

[8] In this case we take $A_*(s)$ to be TI; if otherwise selected, $A_*(s,t)$ should be slowly TV since the speed of variation of its coefficients affects the range of μ for which closed-loop stability can be guaranteed.

where $\Delta(s,t)$ is a PDO of degree at most $2n-2$, with smooth, UB coefficients $O(\mu)$. ▽▽

Proof: In Appendix V.

The results of Theorem 5.14 can be extended to a wider class of slowly TV plants and a wider class of control laws. For example, Assumption 5.12 may be relaxed as

5.15 Assumption: $\|\dot{\Theta}_p\|_\infty \leq \mu$. ■

5.16 Theorem: *Suppose that for the LTV plant (5.1), Assumptions 5.15, 3.1, 5.13 and 3.3 are satisfied except that in 3.1 the plant parameters are only required to be Lipschitz continuous. Also suppose that an LTV controller, of constant order and with a UB parameter vector $\Theta_c(t)$ satisfying $\|\dot{\Theta}_c\|_\infty \leq \mu$,[9] is designed so that the frozen closed loop is ES with rate at most $-a_*$ for all $t \in \mathbf{R}_+$. Then there exists a constant $\mu_2 > 0$ such that $\forall\, \mu \in [0, \mu_2)$,*

1. *the closed-loop system is ES and, therefore, BIBS stable;*
2. *the exponential rate of decay the closed-loop state transition matrix depends on a_* and the value of μ.*

▽▽

Proof: Immediate from Lemma 2.42. Also notice that, in view of Theorem 5.10, it is quite straightforward to further relax the conditions of the theorem to hold in a piecewise sense and on the average. □□

5.17 Remark: Theorem 5.16 allows the construction of a large class of stabilizing controllers for slowly TV plants, including most of the controllers that can be designed using LTI techniques. For example, an LTI PPC, or a Linear Quadratic Gaussian Regulator with guaranteed stability margin (see [Kai.80]), satisfies the above conditions[10] and can be used as PW designs. Furthermore, under Assumption 4.14, a PW MRC also belongs to the same class as a special case of a PPC. Note that a more general and quantitative version of these results has been given in [S.A.91] where the properties of the closed-loop TV sensitivity operators are approximated by the properties of the corresponding operators of the frozen closed-loop plant. ▽▽

5.18 Remark: In the special case of an IMP/PPC design we may select the PDO $P(s,t)$, entering in equations (5.14) and (5.15), in a PW sense (PW IMP) and then perform a TV PPC design for the augmented plant. This approach requires less restrictive and easier to check assumptions while preserving the stabilizing properties of the TV PPC for arbitrarily fast TV plants (see Corollary 5.7). Moreover, it has the advantage that even if Assumption 5.8 is not satisfied, small tracking errors can be achieved for slowly TV plants.

For example, assuming that $N_p(s,t)$, $\Lambda(s)$ are strongly left coprime PDO's in $\mathbf{R}_+$,[11] we can solve

$$\Lambda(s)\hat{X}(s,t) - N_p(s,t)\hat{P}(s,t) = D_p(s,t)Q(s) \tag{5.20}$$

instead of (5.15), by solving a system of linear algebraic equations. Next the TV PPC parameters are calculated as in Corollary 5.7 with $P(s,t), X(s,t)$ being replaced by $\hat{P}(s,t), \hat{X}(s,t)$ respectively. In this case the tracking error becomes

$$\begin{aligned} e \;=\; & N_2(s,t)\bar{A}_*^{-1}(s)\hat{X}(s,t)Q^{-1}(s)\Lambda(s)[r] \\ & + N_2(s,t)\bar{A}_*^{-1}(s)[\Lambda(s)\hat{X}(s,t) - \hat{X}(s,t)\Lambda(s)]Q^{-1}(s)[r] \end{aligned} \tag{5.21}$$

[9] Such a condition holds if, for example, the controller parameter vector Θ_c is a Lipschitz continuous function of the plant parameter vector Θ_p.

[10] Their order is constant and, under Assumption 5.13, their parameters are Lipschitz continuous functions of the plant parameters.

[11] Note that for slowly TV plants, the strong pointwise coprimeness of $N_p(s,t)$ and $\Lambda(s)$ guarantees the strong left coprimeness of the PDO's $N_p(s,t)$, $\Lambda(s)$.

The coefficients of the PDO $[\Lambda(s)\hat{X}(s,t) - \hat{X}(s,t)\Lambda(s)]$ depend on the derivatives of the plant parameters and, hence, the contribution of the last term of (5.21), for a UB reference input, is $O(\mu)$-small if the plant is slowly TV. However, the PPC objective, i.e., $\bar{A}_*^{-1}(s)$ is the closed-loop PIO, is still satisfied for slow and fast TV plants.

A similar result can also be established by using the conventional approach for SISO LTI plants, whereby $\Lambda^{-1}(s)$ is included in the forward path of the loop and the PPC is designed for an augmented plant. Needless to say, in both cases, the exponential stability of the closed-loop system is guaranteed by the PPC design, for arbitrarily fast variations should the TV PPC be used or, for sufficiently slow variations with the PW PPC. Finally, with this approach, an $O(\mu)$ tracking error is also obtained for slowly TV plants with Lipschitz continuous parameters. The details of these statements, however, are omitted as completely analogous to Theorem 5.16. ▽▽

5.7 Examples

The following examples demonstrate the design principles and properties of the TV and PW PPC for LTV plants. For simplicity we drop the argument t in the expressions of the various TV parameters.

5.19 Example: *TV and PW PPC Design.* Let us consider the second order plant

$$\frac{d^2}{dt^2}y_p + \frac{d}{dt}(a_1 y_p) + a_2 y_p = \frac{d}{dt}u_p + bu_p \tag{5.22}$$

where a_1, a_2, b are the TV parameters of the plant and the control objective is to design a controller that makes the closed-loop PIO equal to $(s^3+6s^2+11s+6)^{-1}$. According to the previously presented analysis, we realize a TV PPC law, choosing $F=-2$ and $q=1$, as follows:

$$u_p = p_1(s+2)^{-1}[u_1] + p_3 u_1 \ ; \ u_1 = p_2(s+2)^{-1}[u_1] + (r - y_p) \tag{5.23}$$

where p_1, p_2, p_3 are the controller parameters to be determined. From (5.23) the plant input can also be written as

$$u_p = (s\psi_2 + \psi_3)(s+\psi_1)^{-1}[r - y_p] \ ; \tag{5.24}$$

$$\psi_1 = 2 - p_2 \ ; \ \psi_2 = p_3 \ ; \ \psi_3 = p_1 + 2p_3 - \dot{p}_3 \tag{5.25}$$

Combining (5.22) with (5.24) we obtain the PIO of the closed loop

$$\left[(s^2 + sa_1 + a_2)(s+\psi_1) + (s+b)(s\psi_2 + \psi_3)\right]^{-1} \tag{5.26}$$

which is to be made equal to $[s^3+6s^2+11s+6]^{-1}$. Thus, we obtain the following set of equations that ψ_1, ψ_2, ψ_3 must satisfy:

$$\begin{pmatrix} 1 & 1 & 0 \\ a_1 & b & 1 \\ a_2 & -\dot{b} & b \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \begin{pmatrix} 6 - a_1 \\ 11 - a_2 + \dot{a}_1 \\ 6 + \dot{a}_2 \end{pmatrix} \tag{5.27}$$

or, in a compact form with obvious notation,

$$S_L(t)\psi(t) = A(t) \tag{5.28}$$

Using Cramer's rule, and letting $A = (A_1, A_2, A_3)^\top$ we obtain the solution for ψ

$$\begin{aligned} \psi_1 &= \frac{A_1(b^2 + \dot{b}) - A_2 b + A_3}{\det[S_L(t)]} \\ \psi_2 &= \frac{A_1(a_2 - a_1 b) + A_2 b - A_3}{\det[S_L(t)]} \\ \psi_3 &= \frac{-A_1(a_1\dot{b} + ba_2) + A_2(a_2 + \dot{b}) + A_3(b - a_1)}{\det[S_L(t)]} \end{aligned} \tag{5.29}$$

where, for the solution (5.29) to exist and be bounded $\det[S_L(t)] = b^2 + \dot{b} - a_1 b + a_2$ must be bounded away from zero, for all $t \geq 0$ (Left Coprimeness condition). From (5.25), we obtain the desired controller parameters as

$$p_1 = \psi_3 - 2\psi_2 + \dot{\psi}_2 \ ; \ p_2 = 2 - \psi_1 \ ; \ p_3 = \psi_2 \tag{5.30}$$

Thus, the response of the plant (5.22) with the controller (5.23) and the parameters (5.30), (5.29) is

$$y_p = r - (s + 2 - p_2)(s^3 + 6s^2 + 11s + 6)^{-1}(s^2 + sa_1 + a_2)[r] \tag{5.31}$$

and, hence, the control objective is satisfied exactly with no restrictions on the speed of variation of the plant parameters, other than being finite.

In contrast to the above solution, the design of a PW PPC proceeds by 'freezing' the values of a_1, a_2, b in (5.26) at each time instant [Kre.86, M.G.88], i.e., by taking their derivatives to be identically zero. Thus, the parameters ψ_1, ψ_2, ψ_3 are calculated by solving

$$\begin{pmatrix} 1 & 1 & 0 \\ a_1 & b & 1 \\ a_2 & 0 & b \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \begin{pmatrix} 6 - a_1 \\ 11 - a_2 \\ 6 \end{pmatrix} \tag{5.32}$$

Furthermore, for a pointwise realization of the PPC we select

$$p_1 = \psi_3 - 2\psi_2 \quad ; \quad p_2 = 2 - \psi_1 \quad ; \quad p_3 = \psi_2$$

A comparison with (5.25) shows that the effective controller parameters ψ_i are now ψ_1, ψ_2, $\psi_3 + \dot{\psi}_2$. Thus, using the PW PPC, the PIO (5.26) becomes

$$[s^3 + 6s^2 + 11s + 6 - s(\dot{a}_1 - \dot{\psi}_2) - \dot{a}_2 - \dot{b}\psi_2 + b\dot{\psi}_2]^{-1} \tag{5.33}$$

From (5.33) it becomes apparent that, in general, this solution cannot satisfy the control objective exactly, unless the plant is TI. Furthermore, the closed-loop stability cannot be guaranteed unless the plant is slowly TV, i.e., $\dot{a}_1$, $\dot{a}_2$, $\dot{b}$ are small which, together with the assumed strong nonsingularity of the Sylvester matrix, implies that $\dot{\psi}_i$ is also small. ▽▽

5.20 Simulations: Next, we assign some numerical values for the plant parameters a_1, a_2, b and calculate the corresponding PPC parameters. Let

$$a_1 = 20 + 12 \sin \mu t \ ; \ a_2 = 6 \cos \mu t \ ; \ b = -1 \tag{5.34}$$

where μ is a positive constant which determines the speed of variation of a_1, a_2. Then $\det[S_L(t)] = 21 + 12 \sin 2t + 6 \cos 2t \geq 7.5836 \ \forall t$ and the TV PPC parameters are obtained, in a straightforward way, from (5.29) and (5.30). Moreover, the PW PPC parameters are similarly calculated from (5.32) by observing that, for this example, the PW and left TV Sylvester matrices are the same since $\dot{b} = 0$.

In Fig. 5.4 the response of the closed loop system during regulation ($r = 0$) is shown for the TV and PW PPC. Notice that due to the 'small' value of μ, the PW design is able to preserve the closed-loop stability.

On the other hand, while a larger value of μ does not affect the closed-loop stability for the TV PPC, it is likely to cause the failure of a PW PPC. Letting $\mu = 2$ and using the TV PPC, exact regulation is achieved, as shown in Fig. 5.5. Using the PW PPC, however, the regulation response of the plant becomes unbounded, as shown in Fig. 5.6. In fact, it can be shown (via Floquet analysis) that, for this example, the PIO (5.33) corresponding to the closed-loop with the PW PPC, is unstable. ▽▽

5.21 Example: *TV IMP/PPC Design.* Let us now consider the case where, in addition to the closed-loop stability, the control objective is that the output of the plant (5.22) should track constant reference

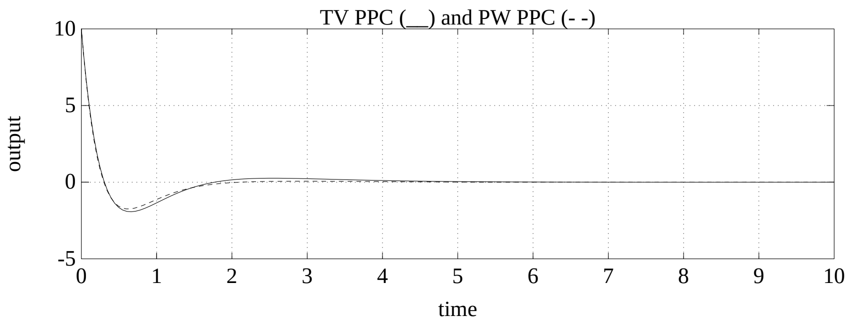

Figure 5.4: Slowly TV plant ($\mu = 0.5$): Exact asymptotic regulation with the TV and PW PPC.

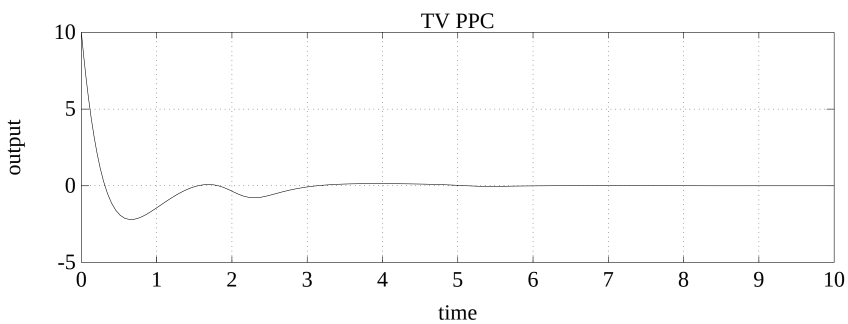

Figure 5.5: Fast TV plant ($\mu = 2$): Exact asymptotic regulation with the TV PPC.

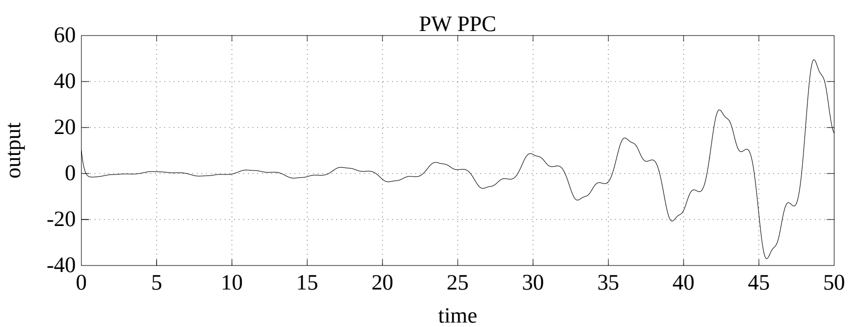

Figure 5.6: Fast TV plant ($\mu = 2$): Unbounded response obtained with the PW PPC during regulation.

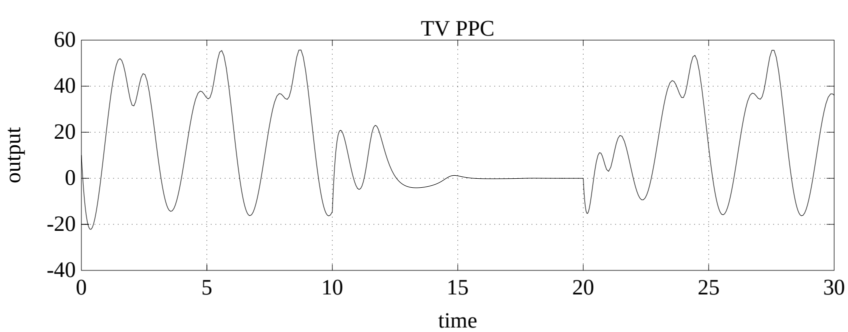

Figure 5.7: Fast TV plant ($\mu = 2$): Poor tracking of step reference inputs with the TV PPC.

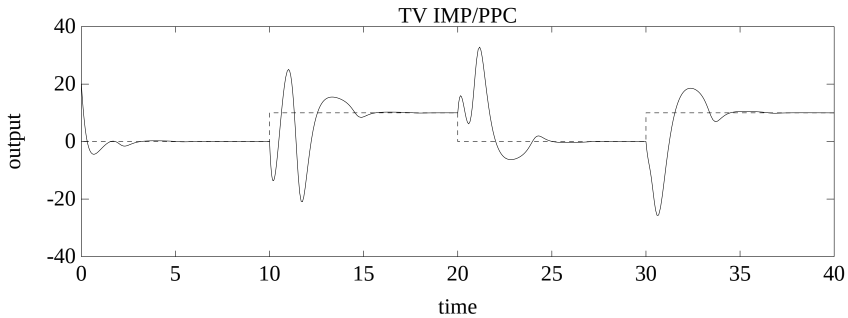

Figure 5.8: Fast TV plant ($\mu = 2$): Exact asymptotic tracking of step reference inputs with the TV IMP/PPC.

signals. In general, the performance of a simple (TV or PW) PPC scheme with respect to such an objective, may be very poor. This is demonstrated in Fig. 5.7 where we use the TV PPC of the previous example (with $\mu = 2$) to track a square wave reference input with values alternating between 0 and 10.

In order to enhance the tracking performance of the TV PPC we employ the results presented in Section 5.4 to design a TV IMP/PPC. Thus, according to Corollary 5.7, we take $\Lambda^{-1}(s) = s^{-1}$ and design the plant input as

$$u_p = -\rho y_p + \bar{u}_p \ ; \ \bar{u}_p = (s\psi_1 + \psi_2)(s + \psi_3)^{-1}[r - y_p] \ ; \tag{5.35}$$

where ρ is a time-varying gain and $\bar{u}_p$ is realized as in the TV PPC case (eqns. (5.23), (5.24)). We observe that in this case, we need not increase the order of the controller since $\deg[\Lambda(s)] = 1 \leadsto \deg[Q(s)] = 0$. From Corollary 5.7 we have that ρ should satisfy

$$s^2 + sa_1 + a_2 + (s + b)\rho = (s + x)s \tag{5.36}$$

for some x. Performing the calculations in (5.36) we obtain

$$\dot{\rho} = -b\rho - (a_2 + \dot{a}_1) \ \ ; \ \ x = a_1 + \rho \tag{5.37}$$

At this point we assume that we can find a bounded function of time, ρ, s.t. (5.37) is satisfied. Note that in the LTI case ($\dot{\rho} = 0$) this assumption reduces to $b \neq 0$, i.e., the plant zero should not be a zero of the internal model $\Lambda(s)$. The rest of the TV IMP/PPC design is done as in the PPC case with $[s + (a_1 + \rho)]s$ in the place of $[s^2 + sa_1 + a_2]$. The final TV IMP/PPC design guarantees the closed-loop internal stability as well as the exact tracking of constant (step) reference inputs. ▽▽

5.22 Simulations: Using the numerical values of Example 5.19, with $\mu = 2$, we obtain

$$\rho = -12\sin 2t + 6\cos 2t \ \ ; \ \ x = 20 + 6\cos 2t \tag{5.38}$$

and the plant PIO, for which the TV PPC should be designed, is now $[s^2 + s(20 + 6\cos 2t) + (12\sin 2t)]^{-1}$ and the calculation of the gains p_1, p_2, p_3 is performed in the same fashion as in Example 5.19, by making the following substitutions:

$$\begin{aligned} a_1 &\longleftarrow 20 + 6\cos 2t \\ a_2 &\longleftarrow 12\sin 2t \end{aligned} \tag{5.39}$$

The exact asymptotic tracking of step reference inputs is demonstrated in Fig. 5.8, where the output of the closed-loop plant is required to follow a square wave reference input. ▽▽

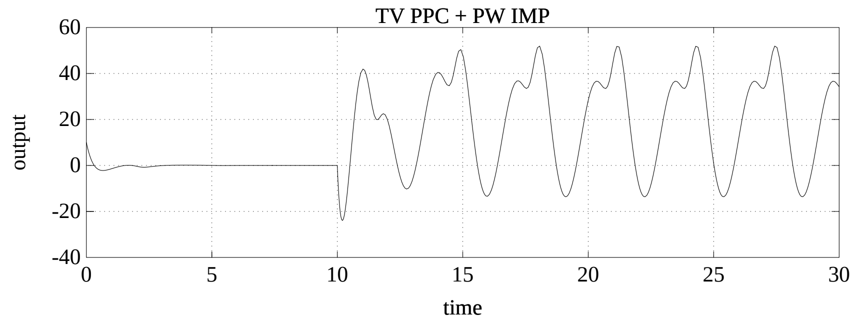

Figure 5.9: Fast TV plant ($\mu = 2$): Bounded response but poor tracking of step reference inputs with the TV PPC + PW IMP.

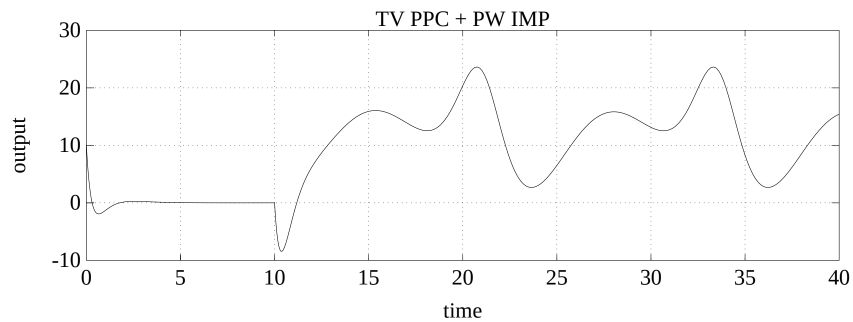

Figure 5.10: Slowly TV plant ($\mu = 0.5$): Somewhat improved tracking of step reference inputs with the TV PPC + PW IMP.

5.23 Example: *TV PPC + PW IMP Design.* Finally, to illustrate the modified IMP design mentioned in Section 5.6, let us consider Example 5.21 again and take $u_p = -\rho y_p + \bar{u}_p$. From (5.22) we obtain

$$[s^2 + sa_1 + a_2 + (s+b)\rho][y_p] = (s+b)[\bar{u}_p]$$

We may now choose ρ to introduce the internal model as a factor of the plant PIO in a PW sense. For example, assuming $b \neq 0$, we may take $\rho = -a_2/b$ (other choices are also possible) which yields the following description for the modified plant

$$[s^2 + s(a_1 + \rho)][y_p] = (s+b)[\bar{u}_p]$$

With this choice, $L(s) = s$ becomes a left factor of the plant PIO. Next, the control law $\bar{u}_p$ is designed as a TV PPC for the modified plant, i.e., for the plant $[s^2+s(a_1+\rho)]^{-1}[s+b]$. The resulting controller guarantees the closed-loop stability irrespective of the speed of the plant parameter variations and $O(\mu)$ tracking error to step reference inputs.

This is demonstrated in Figs. 5.9, 5.10 and 5.11. In the first of these figures $\mu = 2$, i.e., the plant parameters are fast TV; the closed-loop is shown to have bounded response but tracking is very poor.[12] In the last two figures, $\mu = 0.5,\ 0.2$ respectively; the tracking of a step input improves considerably as the speed of the plant parameters decreases. $\nabla\nabla$

APPENDIX V

[12] Step reference input; 0-10 transition at $t = 10$.

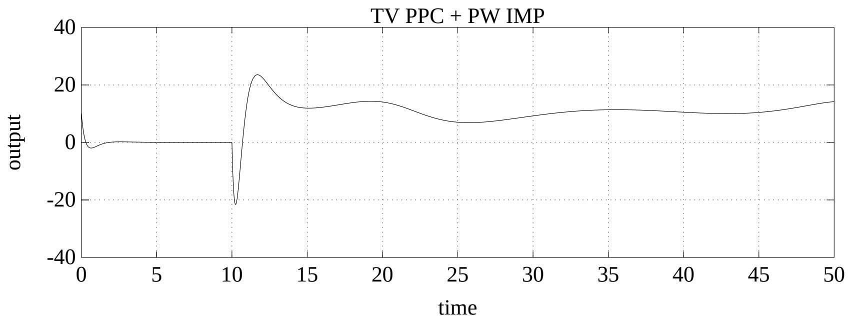

Figure 5.11: Slowly TV plant ($\mu = 0.2$): Considerably improved tracking of step reference inputs with the TV PPC + PW IMP.

Proof of Corollary 5.7:

For the first part of the Corollary we must show that D_pQ and N_p are strongly left coprime PDO's in $\mathbf{R}_+$. From Definition 2.12, it suffices to show that there exist PDO's X, Y, X_0, Y_0 with smooth UB coefficients, such that

$$D_pQX + N_pY = 1 \quad ; \quad D_pQX_0 + N_pY_0 = 0.$$

Since D_p and N_p are strongly left coprime in $\mathbf{R}_+$, the PDO's D_R, N_R, corresponding to the right coprime factorization of the plant, exist and have smooth UB coefficients and satisfy $D_pN_R - N_pD_R = 0$. Further, since Q and N_R are strongly left coprime in $\mathbf{R}_+$, there exist PDO's X_q, Y_q, X_R, Y_R with smooth UB coefficients such that

$$QX_q + N_RY_q = 1 \quad ; \quad QX_R + N_RY_R = 0.$$

Hence, $D_pQX_R + D_pN_RY_R = D_pQX_R + N_pD_RY_R = 0$. Next, let X_1, Y_1 be the PDO's satisfying $D_pX_1 + N_pY_1 = 1$ which, by the strong coprimeness of D_p and N_p, exist and have smooth UB coefficients. Since Q and N_R are strongly left coprime in $\mathbf{R}_+$, there exist PDO's X, W with smooth UB coefficients, such that $QX - N_RW = X_1$. Define the PDO $Y = Y_1 - D_RW$. It follows that $D_pN_RW - N_pD_RW = 0$ and $D_p(X_1 + N_RW) + N_p(Y_1 - D_RW) = 1$ from which we obtain $D_pQX + N_pY = 1$. Letting $X_0 = X_R$ and $Y_0 = D_RY_R$ it follows from Definition 2.12 that the PDO's D_pQ and N_p are strongly left coprime in $\mathbf{R}_+$.

The above result guarantees the existence of two PDO's $\tilde{N}_1, \tilde{N}_2$ with smooth UB coefficients such that $D_pQ\tilde{N}_2 + N_p\tilde{N}_1 = \bar{A}_*$. Letting $N_1 = \tilde{N}_1 - P\tilde{N}_2$, we obtain equation (5.14). The rest of the proof of part (a.) follows from Lemma 2.35 using similar arguments as in Theorems 4.10 and 5.5.

For part (b.) and after some straightforward calculations we obtain the following equation for the tracking error $e = r - y_p$

$$e = N_2(s,t)\left[X(s,t)\Lambda(s)\tilde{N}_2(s,t) + N_p(s,t)N_1(s,t)\right]^{-1} X(s,t)\Lambda(s)Q^{-1}(s)[r] \tag{5.40}$$

where, from Assumption 5.8,

$$X(s,t)\Lambda(s) = D_p(s,t)Q(s) + N_p(s,t)P(s,t).$$

Since $\Lambda(s)Q^{-1}(s)[r] = Q^{-1}(s)\Lambda(s)[r] = Q^{-1}(s)[0]$ we get

$$e = N_2(s)\bar{A}_*^{-1}(s)X(s)Q^{-1}(s)[0] \tag{5.41}$$

Consequently, by part (a.) the closed-loop system is ES and the tracking error decays to zero exponentially fast. □□

Proof of Theorem 5.14:

Note that, by Lemma 2.41, either Assumption 3.2 or its pointwise version 5.13 can be used, since the latter implies the former for $\mu \in [0, \mu_{20})$ and some $\mu_{20} > 0$. For the PW PPC, the coefficients of the controller PDO's $M_i(s,t)$ are obtained by a pointwise solution of the corresponding Diophantine equation (see Lemma 5.1) i.e.,

$$D_p(s,t) \star M_2(s,t) + N_p(s,t) \star M_1(s,t) = A_*(s)$$

where $(\star)$ denotes a pointwise operation. Hence, from property P4 of the PDO's, it follows that $M_i(s,t)$ satisfy

$$D_p(s,t)M_2(s,t) + N_p(s,t)M_1(s,t) = A_*(s) + \Delta_1(s,t) \qquad (5.42)$$

where $\Delta_1(s,t)$ is a PDO of degree $2n-2$, with coefficients $O(\mu)$. Furthermore, since the controller parameters $p_i(t)$ are found by pointwise calculations from $M_i(s,t)$, it follows that the realized control law corresponds to PDO's $\hat{M}_i(s,t)$ such that the PDO's $\hat{M}_i(s,t) - M_i(s,t)$ have coefficients $O(\mu)$ and $\hat{M}_2(s,t) - M_2(s,t)$ is of degree $n-2$. Thus, the actual closed-loop system has PIO

$$D_c^{-1}(s,t) = [D_p(s,t)\hat{M}_2(s,t) + N_p(s,t)\hat{M}_1(s,t)]^{-1} = [A_*(s) + \Delta(s,t)]^{-1} \qquad (5.43)$$

where, again, $\Delta(s,t)$ is a PDO of degree $2n-2$, with coefficients $O(\mu)$. Hence, by Lemma 2.45, the closed-loop system is BIBO stable $\forall \mu \in [0, \mu_2)$, for some $\mu_2 > 0$ (and $\mu_2 \leq \mu_{20}$). Further, the internal stability of the plant and the exponential stability of the closed-loop system follow by using similar arguments as in the proof of Theorem 5.5 where, now, D_c is given by (5.43). □□

Chapter 6

On-Line Parametric Identification

6.1 Introduction

In the previous chapters we established some general controller design techniques which under their respective assumptions and given a complete knowledge of the plant parameters produce a stabilizing controller. In practice, however, such a knowledge is more often than not unavailable, introducing some uncertainty in the dynamical description of the plant. This uncertainty can be in the form of a general dynamical operator (dynamic uncertainty) and/or in the form of parametric uncertainty, that is an error in the parameters of the state-space representation of the (nominal) plant. For both types of uncertainty, a controller designed so as to make the nominal closed-loop plant exponentially stable is also able to guarantee stability in the presence of sufficiently 'small' amounts of uncertainty. For example, such a result may easily be established by employing the small-gain theorem [D.V.75] for dynamic uncertainty whose operator has small gain or Lemma 2.45 for small parametric uncertainty. In the context of TV plants, however, the requirement that the parametric uncertainty is small may be too restrictive and difficult to satisfy. Consider for example a parameter varying as $\sin(w_0 t)$; any small perturbation of the frequency w_0 is sufficient to destroy the knowledge of the parameter within a small or small-in-the-mean-square error.

On the other hand, it is intuitively possible to use a parameter estimation scheme to identify the parameters of the nominal plant on-line, thus reducing a large parametric uncertainty to a level that can be tolerated by the controller. The implementation of this deceivingly simple idea is the subject of the rest of this book. The main theoretical problem, introduced by such an implementation, is the severe nonlinear coupling between the parameter estimator and the control law. This raises questions not only about the stability/boundedness of the closed-loop system but about the existence of solutions of the differential equation describing the closed-loop as well.

In order to provide an answer to these questions, we first need to establish some fundamental properties of parameter estimation algorithms operating in a closed-loop environment. In such an environment the various signals cannot be assumed to be bounded a priori —or, for that matter, even exist at all. Moreover, any convenient arguments on the convergence of the estimated parameters are also absent since the closed-loop signals are not in the disposal of the designer. The derivation of the properties of parameter estimators under such weak conditions is the subject of this chapter. We begin with Section 6.2 where we consider a general affine in the (TV) parameters model and analyze some basic algorithms for the estimation of the TV parameters. The results of Section 6.2 are subsequently employed in Section 6.3 where we discuss the parametric identification of the I/O operator of a general LTV plant. Finally, we present a simple example to illustrate the main ideas of this chapter.

6.2 Affine Parametric Models and Estimation of TV Parameters

Let us consider the parametric model,

$$y(t) = w^{\top}(t)\theta_*(t) + \eta(t) \tag{6.1}$$

where $y : [t_0, t_0 + T] \mapsto \mathbf{R}$, $w : [t_0, t_0 + T] \mapsto \mathbf{R}^n$ are signals available for measurement, $\theta_* : \mathbf{R}_+ \mapsto \mathbf{R}^n$ is a vector of unknown parameters and $\eta : [t_0, t_0 + T] \mapsto \mathbf{R}$ is an unknown signal which is typically due to modelling error effects or noise in (6.1).

Parametric models of the form (6.1) have been extensively studied in the identification of LTI plants, where θ_* is constant (e.g., see [S.B.89, G.S.84]). As we have shown in Chapter 2, the parametric model (6.1) is also applicable in the case of LTV plants, e.g., see equation (3.21) where θ_* is a vector of the PDO coefficients of the plant I/O operator in the P_L form and η is a swapping term depending on $\dot{\theta}_*$. In this section, our objective is to develop and study parameter estimators or adaptive laws to estimate $\theta_*(t)$ in (6.1). For this purpose, we assume that

6.1 Assumption: *θ_* is UB and piecewise Lipschitz continuous on $\mathbf{R}_+$ with piecewise UB derivative; we use n_I to denote the number of points of discontinuity of θ_* —always of the first kind— in an interval $I \subseteq [t_0, t_0 + T]$;* ∎

6.2 Assumption: *w, η (and therefore y) are UB and piecewise continuous on $[t_0, t_0 + T]$; in this context 'UB' also denotes that the bounds are independent of T and t_0;* ∎

6.3 Assumption: *a bounded, convex set $\mathcal{M}(t)$ with smooth boundary and such that $\theta_*(t) \in \mathcal{M}(t)$, $\forall\, t \in \mathbf{R}_+$, is known a priori. For simplicity, we assume that the set $\mathcal{M}(t)$ is a ball in $\mathbf{R}^n$, i.e.,*

$$\mathcal{M}(t) = \mathcal{M} = \{\theta \in \mathbf{R}^n : |\theta - \theta_c| \leq M_0\}$$

where the center θ_c and the radius M_0 are constant. ∎

Assumption 6.2 requires w, η and y to be UB on $[t_0, t_0 + T]$ with bounds that are independent of t_0 and T. In most identification problems of LTV plants parametrized by (6.1), such an assumption is satisfied by considering either stable LTV plants or plants which are stabilized by a fixed (non-adaptive) controller. On the other hand, if Assumption 6.2 fails, it is often possible to rewrite (6.1) as

$$\bar{y}(t) = \bar{w}^{\top}(t)\theta_*(t) + \bar{\eta}(t)$$

where $\bar{x}$ denotes the *normalized signal* x, i.e., $\bar{x}(t) = x(t)/m(t)$ and $m(t)$ is a suitable normalization signal selected so that $\bar{w}, \bar{\eta}$ and $\bar{y}$ satisfy Assumption 6.2. This situation typically arises in an adaptive control setup where the closed-loop signals may not be assumed to be UB a priori. Normalization is therefore a useful tool which allows us to extend any results developed for the parametric model (6.1) to such cases. A more detailed discussion on the design of a normalization signal is given later in this and the next chapter.

In Assumption 6.3, the parameter M_0 used in the definition of the set $\mathcal{M}$ is a measure of the size of the parametric uncertainty in $\theta_*(t)$. With a little additional effort both M_0 and θ_c can be allowed to vary with time. For example, under a similar formulation, it is possible to admit generalized ellipsoids as parametric uncertainty sets, i.e.,

$$\mathcal{M}(t) = \{\theta \in \mathbf{R}^n : |M(t)[\theta - \theta_c(t)]| \leq 1\}$$

where $M(t)$ is a known, strongly nonsingular, positive definite matrix $\forall\, t \in \mathbf{R}_+$, $\theta_c(t)$ is a known vector and M, θ_c are Lipschitz continuous and UB on $\mathbf{R}_+$. At this stage, we do not consider such a generalization, as it unnecessarily complicates the presentation without offering any significant improvement of the results; it may be useful, however, in subsequent studies.

Furthermore, the vector norm in the definition of the set $\mathcal{M}$ can be chosen as the most convenient one for the particular problem although norms whose unit balls have non-smooth boundaries introduce some additional difficulties in the estimator construction. In our discussion we avoid this rather minor issue by considering only two vector norms, namely the $|\cdot|_2$ and the $|\cdot|_\infty$ which are a 'natural' selection in most applications. For example, $|\cdot|_2$ would be less conservative if the uncertainty in $\theta_*(t)$ is specified in terms of the radius of a sphere while $|\cdot|_\infty$ would be more appropriate if it is specified in terms of the minimum and maximum value of its components. Notice that in this case, the estimation problem can be treated component-wise with interval constraints and thus satisfy the smooth boundary assumption; an example of this is presented later.

We may now develop an adaptive law to estimate θ_* in (6.1) according to the following procedure:

Let $\theta(t)$ be the estimate of $\theta_*(t)$ at time t. Then the estimated value of y at time t, based on the estimate $\theta(t)$, is

$$\hat{y}(t) = w^\top(t)\theta(t) \tag{6.2}$$

The *estimation error*

$$\epsilon_1 = \hat{y} - y = w^\top\theta - y \tag{6.3}$$

is therefore a measure of the 'quality' of estimation in the sense of (6.1), that is, how well is the partially unknown parametric model (6.1) approximated by (6.2). Substituting (6.1) and (6.2) in (6.3), it follows that the estimation error is expressed as

$$\epsilon_1 = w^\top\phi - \eta$$

where $\phi \stackrel{\Delta}{=} \theta - \theta_*$ is the parameter error. In other words, ϵ_1 contains information about the parameter error, in an inner product form, corrupted by the noise term η. An adaptive law to update the estimate θ is designed by using the gradient projection method to minimize the cost

$$J(\theta) = \epsilon_1^2 = (w^\top\theta - y)^2$$

with respect to θ and subject to the constraint $\theta \in \mathcal{M}$. Such an update law has the form

$$\dot{\theta} = \mathcal{P}\left(-\gamma\epsilon_1 w\right) \;\; ; \;\; \theta(t_0) \in \mathcal{M} \tag{6.4}$$

where $\gamma > 0$ is a constant gain referred to as the *adaptation gain* and, for simplicity, is taken as scalar and $\mathcal{P}$ denotes a projection operator, designed to guarantee $\theta \in \mathcal{M}$.

Essentially, the operator $\mathcal{P}$ is the identity when θ is in the interior of $\mathcal{M}$ and projects $-\gamma\epsilon_1 w$ on the tangent hyperplane at $\theta(t)$ when the latter is on the boundary of $\mathcal{M}$ and the vectorfield points towards the exterior of $\mathcal{M}$ (e.g., see [Ega.79, G.S.84, S.B.89]). Some additional provisions are taken so that the vectorfield in (6.4) is (at least locally) Lipschitz continuous, in order to avoid any problems with the existence and uniqueness of θ, as well as problems in the numerical simulations of the adaptive law. For this purpose, we may slightly increase the size of the set $\mathcal{M}$, i.e., define a boundary region of some small but nonzero thickness, where we make a smooth transition between projected and unprojected vectorfields. The thickness of the boundary region is denoted throughout by ϵ_* which is treated as a small design parameter. In the following examples we illustrate such a design of the projection operator in the two, most frequently encountered, cases where the set $\mathcal{M}$ is specified in terms of an L_∞ and an L_2 vector norm. It should be mentioned that the existence and uniqueness (in the sense of Fillipov) of solutions of differential equations of the form (6.4) have been established in [P.I.91] for a general projection operator. It is therefore theoretically possible to select $\epsilon_* = 0$. In practice, however, such a choice may cause numerical problems due to the discontinuous vectorfield in the differential equation with projection. For this reason, in the following we assume that $\epsilon_* > 0$.

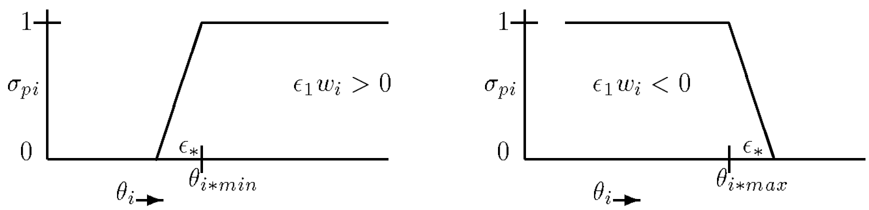

Figure 6.1: The functions σ_{pi}.

6.4 Example: When the set $\mathcal{M}$ is defined in terms of the minimum and maximum values of the components of θ_*, say θ_{i*min} and θ_{i*max} respectively, the projection operator $\mathcal{P}$ is simply constructed as a multiplier of the form

$$\mathcal{P} = \text{diag}[\sigma_{pi}]$$

$$\sigma_{pi} = \begin{cases} \max\left\{0, \min\left[1, 1 + \frac{\theta_i - \theta_{i*min}}{\epsilon_*}\right]\right\} & \text{when } \epsilon_1 w_i > 0 \\ \max\left\{0, \min\left[1, 1 + \frac{\theta_{i*max} - \theta_i}{\epsilon_*}\right]\right\} & \text{when } \epsilon_1 w_i < 0 \\ 1 & \text{otherwise} \end{cases} \tag{6.5}$$

where ϵ_* is an arbitrary, small positive constant whose purpose is to ensure the Lipschitz continuity (in θ) of $\sigma_{pi}\epsilon_1 w_i$, something that can be shown using Assumptions 6.1–6.3. Pictorially, the functions σ_{pi} are shown in Fig. 6.1. $\nabla\nabla$

6.5 Example: A similar construction of the projection operator $\mathcal{P}$ is also possible when the set $\mathcal{M}$ is defined in terms of the Euclidean norm of θ_*. In this case the set $\mathcal{M}$ is of the general form

$$\mathcal{M} = \{\theta \in \mathbf{R}^n : |\theta - \theta_c|_{2,M} \leq 1\}$$

where $|\theta|_{2,M}$ denotes the weighted Euclidean norm $\left(\theta^\top M\theta\right)^{1/2}$ and M is a symmetric positive definite matrix. In other words, $\mathcal{M}$ is a generalized ellipsoid in $\mathbf{R}^n$ centered at θ_c.

For such a set, one simple form of a projection operator can be given in terms of a multiplier

$$\mathcal{P} = I - \sigma_p \theta_\perp \theta_\perp^\top$$

where

$$\theta_\perp = \frac{M(\theta - \theta_c)}{|M(\theta - \theta_c)|_2}$$

is the normal vector, pointing outwards, on the surface $|\theta - \theta_c|_{2,M} = constant$,[1]

$$\sigma_p = \begin{cases} \max\left\{0, \min\left[1, \frac{|\theta - \theta_c|_{2,M} - 1}{\epsilon_*}\right]\right\} & \text{when } \epsilon_1 w^\top \theta_\perp < 0 \\ 0 & \text{otherwise} \end{cases} \tag{6.6}$$

and ϵ_* is again an arbitrary, small positive constant. $\nabla\nabla$

The properties of the adaptive law (6.4) are given by the following theorem.

6.6 Theorem: *Consider the parametric model (6.1), satisfying Assumptions 6.1–6.3, and the adaptive law (6.4). Then,*

a. θ, $\dot\theta$ are UB on $[t_0, t_0 + T]$ (θ is within distance ϵ_* from $\mathcal{M}$);

b. there exist constants C_0, C_1, K_ϕ, K_J, independent of T, t_0, such that for any interval $I = [t_I, t_I + T_I] \subseteq [t_0, t_0 + T]$,

[1] The constant used here is the current value of $|\theta - \theta_c|_{2,M}$, i.e., $\theta_\perp$ is the normal vector on the boundary of a scaled ellipsoid, similar to $\mathcal{M}$, that passes through the point θ. Note that a more efficient but more complicated type of projection would be to take $\theta_\perp$ to be the vector of minimum distance between θ and $\mathcal{M}$.

1. $\displaystyle\int_{t_I}^{t_I+T_I} \epsilon_1^2(t)\,dt \le C_0 + \int_{t_I}^{t_I+T_I} \eta^2(t)\,dt + \frac{K_\phi}{\gamma}\int_{t_I}^{t_I+T_I} |\dot{\theta}_*^s(t)|_i\,dt + \frac{K_J}{\gamma} n_I$

2. $\displaystyle\int_{t_I}^{t_I+T_I} |\dot{\theta}(t)|^2\,dt \le \gamma^2 C_1 \int_{t_I}^{t_I+T_I} \epsilon_1^2(t)\,dt$

where θ_*^s denotes the Lipschitz continuous part of θ_*, i.e., $\theta_* = \theta_*^s + \theta_*^J$, for some piecewise constant (jump) function θ_*^J and $|\cdot|_i$ is the linear functional norm induced by the norm employed in the definition of the set $\mathcal{M}$. ▽▽

Proof: In Appendix VI.

At this point, it is worthwhile to make some observations on the performance of the adaptive law (6.4), as given by Theorem 6.6 and its relation with the various design parameters.

• The adaptive law (6.4) identifies the I/O properties of θ_*, viewed as an operator defined by (6.1), in the mean-square sense. In this sense, the mean-square value of the estimation error is a measure of the quality of identification. Thus, the identification is 'successful' from an I/O point of view, provided that θ_* varies slowly with time and the noise term η is small.

• The constants used in the theorem depend on the size of parametric uncertainty (M_0) and the gain of adaptation (γ) as follows:

$$C_0 = O(M_0^2/\gamma) \quad ; \quad K_\phi = O(M_0) \quad ; \quad C_1 = ||w||_\infty^2$$

while K_j is of the order of the maximum size of jumps. These relations imply that if the adaptation gain is large enough compared with the speed of variation of θ_*, the mean-square value of the estimation error is of the order of the mean-square value of the noise term η.

• In the special case where $\eta = 0$ and $\theta_* = constant$, the estimation error ϵ_1 is square integrable. It follows that if ϵ_1 is also uniformly continuous, then as $T \to \infty$, $\epsilon_1 \to 0$. This does not imply, however, that for small η and $\dot{\theta}_*$, ϵ_1 remains uniformly small as $T \to \infty$. Instead, properties (2) and (3) allow for the possibility of 'burst' phenomena, whereby the estimation error may attain $O(M_0)$-large values during short time periods. Such phenomena can be avoided in the case of constant parameters by using some a priori information on the L_∞ bound of η and a 'dead-zone' modification in the adaptive law (e.g., see [P.N.82, MGHM.88, G.S.84]). It is not clear though, whether this technique can be transferred to the TV case with similar results.

• Even in the TI case, Theorem 6.6 does not provide any conclusions on the 'strong' convergence of the operator θ to θ_*, i.e., the convergence of $||\theta - \theta_*||$ to zero. Although such a convergence is desirable, it does not seem to be possible unless θ_* is constant and the vector w is *persistently exciting* [And.77]. When the unknown parameter vector θ_* is slowly TV this result can be extended to the parametric model (6.1) within an error that depends on the speed of variation of θ_* [M.G.87].

Analogous results can be obtained with estimators employing the so-called σ-modification [I.K.83]. With this modification, a term $-\sigma(\theta - \theta_c)$ is introduced in the vectorfield of the estimator, providing the necessary component to ensure the boundedness of the parameter estimates, for example

$$\dot{\theta} = -\gamma\epsilon_1 w - \sigma(\theta - \theta_c) \tag{6.7}$$

where σ is a positive constant.

6.7 Theorem: *Consider the parametric model (6.1), satisfying Assumptions 6.1–6.2 and the adaptive law (6.7). Then,*

a. θ, $\dot{\theta}$ are UB on $[t_0, t_0 + T]$ and $\theta(t)$ converges exponentially fast to a residual set

$$\left\{\theta : |\theta(t) - \theta_c|_2 \le \sqrt{\frac{\gamma}{2}}\mathcal{E}_{-\sigma}(t)||(y - w^\top\theta_c)_t||_{2,\sigma}\right\}$$

b. there exist constants C_0, C_1, K_J, independent of T, t_0, such that for any interval $I = [t_I, t_I + T_I] \subseteq [t_0, t_0 + T]$,

1. $$\int_{t_I}^{t_I+T_I} \epsilon_1^2(t)\,dt \le C_0 + \int_{t_I}^{t_I+T_I} \eta^2(t)\,dt \ldots$$
$$+\frac{1}{2\gamma}\int_{t_I}^{t_I+T_I} |\sqrt{\sigma}(\theta_*(t)-\theta_c) + \frac{1}{\sqrt{\sigma}}\dot{\theta}_*^s(t)|_2^2\,dt + \frac{K_J}{\gamma} n_I$$

2. $$\left[\int_{t_I}^{t_I+T_I} |\dot{\theta}(t)|_2^2\,dt\right]^{1/2} \le \gamma\left[C_1\int_{t_I}^{t_I+T_I} \epsilon_1^2(t)\,dt\right]^{1/2} \ldots$$
$$+\sigma\left[\int_{t_I}^{t_I+T_I} |\theta(t)-\theta_c|_2^2\,dt\right]^{1/2}$$

where θ_*^s denotes the Lipschitz continuous part of θ_*. $\nabla\nabla$

Proof: In Appendix VI.

Such an estimator has the advantage that parameter boundedness is guaranteed without the explicit a priori knowledge of a bound on the parametric uncertainty. However, unlike an estimator using projection, it does not recover any asymptotic identification properties when the parameters become constants. An additional drawback is that the selection of σ and γ involves certain trade-offs since both σ and $1/\sigma$ appear in the various bounds. For example, $\frac{\sigma}{\gamma}|\theta_* - \theta_c|^2$ and $\frac{1}{\gamma\sigma}|\dot{\theta}_*^s|^2$ both appear as perturbations in the mean-square value of the estimation error. And although their contribution can be lessened by increasing the value of the adaptation gain, such an approach has the undesirable effect of increasing the bound of θ as well as the mean-square value of $\dot{\theta}$ and may cause instability in an adaptive control scheme employing (6.7). On the other hand, the properties of the above theorem indicate that this estimator can be particularly effective when the (unknown) parametric uncertainty is small (i.e., $|\theta_* - \theta_c|$ is 'small').

It should be mentioned that another difference between the two techniques discussed above is that using an estimator with projection, the parameters are guaranteed to be UB with considerably weaker assumptions. In fact, if the bounds of the signals w, η depend on T, the projection still guarantees the uniform boundedness of θ, although in this case the constants in the Theorem 6.6 may depend on T. The same statement is not true for an estimator with a σ-modification, in which case, the failure of w, η to be UB results in parameter estimates whose bound may depend on T. Such a property, however, is not exploited any further and in the sequel Assumption 6.2 —or sufficient conditions for it to be satisfied— is employed whenever a parameter estimation algorithm is constructed.

6.8 Remark: It is possible to retain most of the attractive simplicity of the σ-modification and, at the same time, achieve error bounds as in Theorem 6.6 by using the a priori knowledge of M_0 to switch on the σ term only when the magnitude of the parameter estimates is large. In this case, σ is chosen as a Lipschitz function of $|\theta - \theta_c|_2$, e.g.,

$$\sigma = \sigma_0 \max\left\{0, \min\left[1, \frac{|\theta-\theta_c|_2 - M_0}{\epsilon_*}\right]\right\}$$

where σ_0 is a positive constant. This switching σ-modification has the property that, provided $|\theta_* - \theta_c|_2 < M_0$, $\sigma(\theta-\theta_c)^\top(\theta-\theta_*) \ge 0$.[2] Using this property it is straightforward to verify that Theorem 6.7 is still valid and the mean-square value of ϵ_1 satisfies

$$\int_{t_I}^{t_I+T_I} \left[\epsilon_1^2(t) + \frac{2\sigma}{\gamma}(\theta(t)-\theta_c)^\top(\theta(t)-\theta_*(t))\right] dt \le C_0' + \ldots$$
$$+\int_{t_I}^{t_I+T_I} \eta^2(t)\,dt + \frac{K_\phi'}{\gamma}\int_{t_I}^{t_I+T_I} |\dot{\theta}_*^s(t)|\,dt + \frac{K_J'}{\gamma} n_I$$

[2]Otherwise, the switching σ-modification acts effectively as the constant one with similar properties.

where the constants C_0', K_ϕ', K_J' have similar properties as C_0, K_ϕ, K_J of the projection algorithm. Moreover, θ is UB and $(\theta - \theta_c)$ is exponentially decaying to a set bounded (in the 2-norm) by $M_0 + O[\epsilon_*] + O[\sqrt{\frac{\gamma}{\sigma_0}}]$. Hence, the size of this set is essentially independent of the initial conditions $\theta(t_0)$ and, with an appropriate choice of ϵ_* and the ratio γ/σ_0, can be made arbitrarily close to M_0. An attractive property of the switching σ-modification is that it can be used together with a projection to accommodate a priori known time-varying parametric uncertainty (θ_c, M_0) sets, while preserving the properties of the projection algorithms. Note that when $\mathcal{M}(t)$ changes with time, there is a possibility of θ being well outside the projection set at some time instant. In such a case, the term $-\sigma(\theta - \theta_c)$ provides means to bring θ back to $\mathcal{M}(t)$ exponentially fast. $\nabla\nabla$

It is now apparent that the estimators presented in this section have the essential properties required for the successful I/O identification of a slowly TV multiplier θ_*. The quality of the identification is determined by the mean-square value of the estimation error and, as intuitively expected, improves as the speed of variations of θ_* decreases. What remains to be established is that the unknown I/O operator can be described in a form satisfying the assumptions of Theorem 6.6 and 6.7. This problem is studied in the following section.

6.3 Parametric Identification of LTV Plants

In this section we discuss the identification problem of the I/O operator of a LTV plant. Our focus is on parametric identification whereby we perform the identification of the operator by estimating the parameters of a suitable parametric model. Such parametric models for the LTV plants under consideration have been derived in Chapter 3 [e.g., see equations (3.20), (3.21)] and are in or can be transformed to the standard form (6.1). Thus, we can apply the results of the previous section to estimate the parameter vector θ_* in, say, (3.21) which in turn corresponds to the PDO coefficients of the P_L-form of the plant I/O operator. The analysis of such identification schemes and the derivation of error bounds assessing the quality of the parameter estimates as well as the quality of the identified plant I/O operator are the subjects of this section. For reasons of clarity we consider the case of smooth parameter variations first and then generalize the results to the case of non-smooth parameter variations.

6.3.1 Smooth Parameter Variations

Let us consider an LTV plant with a state-space representation given by (3.1) and satisfying Assumptions 3.1–3.3 and suppose that its order n is known. Invoking Lemma 2.32, the plant representation is topologically equivalent to the corresponding observable canonical form and, therefore, it admits an I/O description in terms of PDO's given by

$$y_p = G_p^L(s,t)[u_p] = D_p^{-1}(s,t)N_p(s,t)[u_p]$$

Using Lemma 3.10, the I/O description of the plant becomes

$$y_p = G(s)[u_p\theta_{1*}] + G(s)[y_p\theta_{2*}] \tag{6.8}$$

where $G(s) = q^\top(sI - F)^{-1}$ is a filter selected by the designer and q, F are as specified in that lemma. Since the vector $\theta_* = [\theta_{1*}^\top, \theta_{2*}^\top]^\top : \mathbf{R}_+ \mapsto \mathbf{R}^{2n}$ is directly related to the coefficients of $D_p(s,t)$ and $N_p(s,t)$ by a constant affine transformation depending on q, F, it follows that $\theta_*(t)$ is also smooth and UB.

Further, in order to allow nonzero initial conditions at $t = t_0$, in both the plant and the auxiliary filters $G(s)$, we augment equation (6.8) by an additional exponentially decaying term, as follows

$$y_p = G(s)[u_p\theta_{1*}] + G(s)[y_p\theta_{2*}] + \varepsilon \tag{6.9}$$

where $|\varepsilon(t,t_0)| \le c_0 \exp[-a(t-t_0)]$, c_0 is a constant depending on the size of the initial conditions and $-a$ is the rate of exponential stability of the auxiliary filters (e.g., see proof of Lemma 3.10).

Next, employing the notion of structured parameter variations we assume, without loss of generality, that

$$\theta_*(t) = \Pi(t)\hat{\theta}_*(t)$$

where $\Pi(t)$ is a known (not necessarily square) matrix with smooth UB entries and $\hat{\theta}_*(t)$ is a partially unknown smooth UB vector. Letting $\Pi_1(t)$, $\Pi_2(t)$ be a partition of $\Pi(t)$ such that $\theta_{i*}(t) = \Pi_i(t)\hat{\theta}_*(t)$, equation (6.9) becomes

$$y_p = G(s)[u_p \Pi_1 \hat{\theta}_*] + G(s)[y_p \Pi_2 \hat{\theta}_*] + \varepsilon \tag{6.10}$$

We may now invoke the swapping Lemma 2.59 to express (6.10) in the general form of the parametric model (6.1)

$$\begin{aligned} y_p &= w^\top \hat{\theta}_* - \eta \\ w &= [G(s)\{u_p \Pi_1\} + G(s)\{y_p \Pi_2\}]^\top \\ \eta &= w^\top \hat{\theta}_* - G(s)[u_p \Pi_1 \hat{\theta}_*] - G(s)[y_p \Pi_2 \hat{\theta}_*] \\ &= G(s)\{G'(s)[u_p \Pi_1]\dot{\hat{\theta}}_*\} + G(s)\{G'(s)[y_p \Pi_2]\dot{\hat{\theta}}_*\} \\ G'(s) &= (sI - F)^{-1} \end{aligned} \tag{6.11}$$

In order to employ Theorem 6.6 or 6.7, we need to ensure the boundedness of the regressor vector w and swapping term η in the parametric model (6.11). One way to achieve this is to normalize both sides of (6.11) by using one of the normalization signals described in Chapter 2. The general form of these signals is $m_p^{1/p}$ where $p \in [1,\infty)$ and m_p is constructed by integrating the following differential equation

$$\dot{m}_p = -p\delta_0 m_p + |QU|^p + q_e \;; \quad m_p(t_0) > 0 \tag{6.12}$$

where Q is a positive definite weighting matrix, $U = [u_p, y_p]^\top$, q_e is a positive constant and δ_0 is such that $(F + \delta_0 I)$ is a Hurwitz matrix. Since the g_{p,δ_0}-gains of G and G' are finite, Lemma 2.56 shows that the ratio $x/m_p^{1/p}$, x being any of the signals w, η, y_p, is UB, for as long as the truncated U_t belongs to L_∞, (implying, of course, that $U_t \in L_p(\delta)$ for $t < \infty$).

Thus, the plant I/O description after normalization becomes

$$\frac{y_p}{m_p^{1/p}} = \frac{w^\top \hat{\theta}_*}{m_p^{1/p}} - \frac{\eta}{m_p^{1/p}} + \frac{\varepsilon}{m_p^{1/p}} \tag{6.13}$$

with the normalized signals being UB and satisfying Assumption 6.2 for as long as the truncated U_t belongs to L_∞. Applying the procedure of Section 6.2 on (6.11) we define the estimation error ϵ_1 by

$$\epsilon_1 = w^\top \hat{\theta} - y_p$$

and the normalized estimation error

$$\frac{\epsilon_1}{m_p^{1/p}} = \frac{w^\top \hat{\theta}}{m_p^{1/p}} - \frac{y_p}{m_p^{1/p}}$$

where $\hat{\theta}(t)$ is the estimate of $\hat{\theta}_*(t)$ at time t. The last equation defines a (normalized) parametric model that satisfies Assumptions 6.1–6.3. Therefore, the results of the previous section are now applicable, providing means to design and analyze adaptive laws to estimate $\hat{\theta}_*$. Of course, once $\hat{\theta}$ becomes available, the estimate $\theta(t)$ of $\theta_*(t)$ is obtained from

$$\theta(t) = \Pi(t)\hat{\theta}(t)$$

Note that the estimation of θ via $\hat{\theta}$ may also be viewed as a Kalman filtering problem where θ is known to satisfy a differential equation $\dot{\theta} = A(t)\theta$ with unknown initial conditions. In this case the matrix $\Pi(t)$ corresponds to the completely or partially known state transition matrix associated with $A(t)$.

Before proceeding with the design and analysis of adaptive laws to estimate $\hat{\theta}_*$, let us summarize at this point the main notational conventions.

- U denotes the I/O vector $[u_p, y_p]^\top$. It is assumed throughout this section that the truncated $(u_p)_t$ belongs to L_∞, for all t in a compact interval $[t_0, t_0+T]$; $T>0$. For the class of plants considered it follows that $(y_p)_t$, $U_t \in L_\infty$ over the same interval; we use the notation

$$U \in L_{\infty,[t_0,t_0+T]}$$

  to signify this assumption. Notice that this assumption does not require u_p or y_p to be in L_∞^e or the plant to be stable in any sense.

- $\phi, \hat{\phi}$ denote the parameter errors $\theta - \theta_*$ and $\hat{\theta} - \hat{\theta}_*$ respectively, related by $\phi(t) = \Pi(t)\hat{\phi}(t)$. Whenever necessary, the matrix Π may be partitioned as Π_i such that $\theta_{i*} = \Pi_i\hat{\theta}_*$. We also denote by $\hat{\theta}_c$ a bias term, possibly zero, being the center of the ellipsoid containing $\hat{\theta}_*$.

- $G(s) = q^\top(sI-F)^{-1}$, $G'(s) = (sI-F)^{-1}$ are frequently used. Also, to simplify the various expressions, we use the shorthand notation G_π, G'_π, G_{θ_*}, G_{θ_c} to signify the operators defined by their I/O pairs:

$$\begin{aligned} G_\pi &: QU \mapsto w \triangleq G(s)[u_p\Pi_1] + G(s)[y_p\Pi_2] \\ G'_\pi &: QU \mapsto G'(s)[u_p\Pi_1] + G'(s)[y_p\Pi_2] \\ G_{\theta_*} &: QU \mapsto G(s)[u_p\theta_{1*}] + G(s)[y_p\theta_{2*}] = y_p \\ G_{\theta_c} &: QU \mapsto G(s)[u_p\Pi_1\hat{\theta}_c] + G(s)[y_p\Pi_2\hat{\theta}_c] \end{aligned}$$

  where Q is a positive definite matrix.

- η and $\hat{\eta}$ are reserved for the 'swapping' terms

$$\begin{aligned} \eta &= w^\top\hat{\theta}_* - G(s)[u_p\Pi_1\hat{\theta}_*] - G(s)[y_p\Pi_2\hat{\theta}_*] \\ \hat{\eta} &= w^\top\hat{\theta} - G(s)[u_p\Pi_1\hat{\theta}] - G(s)[y_p\Pi_2\hat{\theta}] \end{aligned}$$

  which, for absolutely continuous $\theta_*, \hat{\theta}$ and zero initial conditions become

$$\begin{aligned} \eta &= G(s)\{G'(s)[u_p\Pi_1]\dot{\hat{\theta}}_*\} + G(s)\{G'(s)[y_p\Pi_2]\dot{\hat{\theta}}_*\} \\ \hat{\eta} &= G(s)\{G'(s)[u_p\Pi_1]\dot{\hat{\theta}}\} + G(s)\{G'(s)[y_p\Pi_2]\dot{\hat{\theta}}\} \end{aligned}$$

- e_1 is used for the 'identification' error defined by

$$e_1 = G(s)[u_p\theta_1] + G(s)[y_p\theta_2] - y_p$$

  and ϵ_1 for the 'estimation' error

$$\epsilon_1 = w^\top\hat{\theta} - y_p$$

  while ε denotes an exponentially decaying term describing the effect of initial conditions.

With this notation, the block diagram of the identifier structure is shown in Fig. 6.2. It is straightforward to verify that the identification error satisfies

$$e_1 = G(s)[u_p\phi_1] + G(s)[y_p\phi_2] + \varepsilon$$

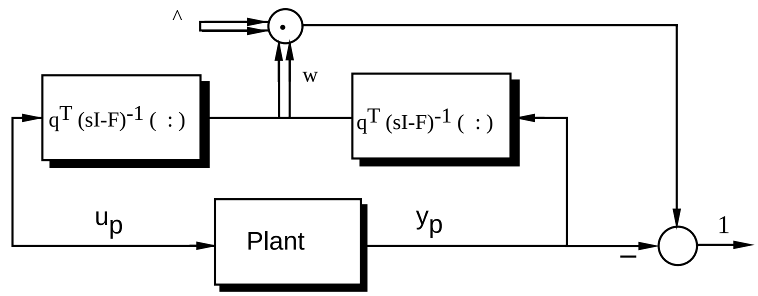

Figure 6.2: Block diagram of the plant identifier.

and hence it is unbiased, in the sense that for $\hat{\theta} = \hat{\theta}_*$, e_1 is decaying to zero exponentially fast. The identification error is therefore appropriate to assess the quality of the parameter estimates interpreted as a part of the plant I/O identification algorithm, i.e., whenever $G(s)[u_p\theta_1] + G(s)[y_p\theta_2]$ is used as an estimate of the plant I/O operator. On the other hand, the estimation error, related to the parameter error via

$$\epsilon_1 = w^\top \hat{\phi} + \eta$$

has the desired for estimation purposes inner product form between the parameter error and a regressor vector (similarly for the normalized estimation error). Notice, however that, unless $\hat{\theta}_*(t)$ is constant, $\hat{\theta} = \hat{\theta}_*$ does not necessarily imply that ϵ_1 is exponentially decaying or even converging to zero.

6.9 Corollary: *Suppose that for an LTV plant satisfying Assumptions 3.1–3.3, $U \in L_{\infty,[t_0,t_0+T]}$ and $\hat{\theta}_*$ satisfies Assumption 6.3. Then, the estimator*

$$\dot{\hat{\theta}} = \mathcal{P}\left(-\gamma \frac{\epsilon_1 w}{m_p^{2/p}}\right) \quad ; \quad \hat{\theta}(t_0) \in \mathcal{M}$$

guarantees that $\hat{\theta}$, $\dot{\hat{\theta}}$ are UB on $[t_0, t_0+T]$ and there exist constants C_0, C_0', C''_0, K_ϕ, independent of T, t_0, such that for any interval $I = [t_I, t_I + T_I] \subseteq [t_0, t_0+T]$

1. $$\int_{t_I}^{t_I+T_I} \left(\frac{\epsilon_1(t)}{m_p^{1/p}(t)}\right)^2 dt \leq C_0 + \frac{K_\phi}{\gamma}\int_{t_I}^{t_I+T_I} |\dot{\hat{\theta}}_*(t)|_i \, dt \ldots$$
$$+ \left[g_{p,\delta}[G] g_{p,\delta_0}[G_\pi'] \left\{ \int_{t_I}^{t_I+T_I} \left(\int_{t_I}^{t} e^{-p(\delta-\delta_0)(t-\tau)} |(\dot{\hat{\theta}}_*(\tau)|^p \, d\tau \right)^{\frac{2}{p}} dt \right\}^{\frac{1}{2}} \right.$$
$$\left. + C_0' \right]^2;$$

2. $$\int_{t_I}^{t_I+T_I} |\dot{\hat{\theta}}(t)|^2 \, dt \leq \left[\gamma g_{p,\delta_0}[G_\pi] \left(\int_{t_I}^{t_I+T_I} \left(\frac{\epsilon_1(t)}{m_p^{1/p}(t)}\right)^2 dt \right)^{1/2} \ldots \right.$$
$$\left. + C''_0 \right]^2;$$

where $\delta > \delta_0$ is such that $F + \delta I$ is a Hurwitz matrix, $K_\phi \leq 4M_0 + 2\epsilon_*$ and the $g_{p,\delta}$-gains of the various operators are evaluated with respect to the underlying vector space norm. ▽▽

Proof: In Appendix VI.

The above corollary gives a quite general description of the mean-squared normalized estimation error in terms of the speed of the unstructured part of the plant parameter variations. Despite its generality, however, the resulting expressions are quite complicated and, perhaps, unnecessarily so. Therefore, we choose at this point to work with the normalization signal m_2 only, yielding more convenient and compact expressions.

Also, in an effort to further simplify the various bounds we introduce a parameter μ as a measure of the 'average' speed of the unstructured parameter variations. That is, we assume that

6.10 Assumption: *There exist constants c, μ such that*

$$\textbf{(a.)} \quad \int_{t_0}^{t_0+T} |\dot{\hat{\theta}}_*(t)|_i \, dt \leq c + \mu T$$

$$\textbf{(b.)} \quad \int_{t_0}^{t_0+T} |\dot{\hat{\theta}}_*(t)|^2 \, dt \leq c + \mu T$$

for all $t_0, T \geq 0$.[3] ∎

We are now in the position to give a simplified and more intuitive, albeit more conservative, version of Corollary 6.9, stated as follows.

6.11 Corollary: *Under the assumptions of Corollary 6.9 and Assumption 6.10, the estimator*

$$\dot{\hat{\theta}} = \mathcal{P}\left(-\gamma \frac{\epsilon_1 w}{m_2}\right) \quad ; \quad \hat{\theta}(t_0) \in \mathcal{M}$$

guarantees that $\hat{\theta}, \dot{\hat{\theta}}$ are UB on $[t_0, t_0 + T]$ ($|\hat{\theta} - \hat{\theta}_c| \leq M_0 + \epsilon_*$); furthermore, there exists a constant C_0 depending on the initial conditions but independent of T, t_0, such that for any interval $I = [t_I, t_I + T_I] \subseteq [t_0, t_0 + T]$

1. $\displaystyle\int_{t_I}^{t_I+T_I} \frac{\epsilon_1^2(t)}{m_2(t)} \, dt \leq C_0 + \left[\Gamma_1^2 + \frac{K_\phi}{\gamma}\right] \mu T_I;$
2. $\displaystyle|\dot{\hat{\theta}}(t)| \leq \gamma \Gamma_2 \frac{|\epsilon_1|}{\sqrt{m_2}} + \varepsilon(t, t_0);$

where $K_\phi, \Gamma_1, \Gamma_2$ are constants depending only on operator gains and the radius of the set $\mathcal{M}$ (their expressions are given in the proof). ▽▽

Proof: For the proof of the corollary we follow the same steps as in Corollary 6.9 except that we also use the Cauchy inequality

$$(x + y)^2 \leq (1 + \epsilon_c)x^2 + (1 + \frac{1}{\epsilon_c})y^2$$

where $\epsilon_c > 0$ is an arbitrary constant ('Cauchy constant') to group the more interesting integral terms together. From Corollary 6.9, we can easily obtain the following estimates for the various bounds

$$K_\phi = 4M_0 + 2\epsilon_* \quad ; \quad \Gamma_1 = (1 + \epsilon_c)\frac{g_{2,\delta}[G] g_{2,\delta_0}[G'_\pi]}{\sqrt{2(\delta - \delta_0)}} \quad ; \quad \Gamma_2 = g_{2,\delta_0}[G_\pi]$$

where $\epsilon_c > 0$ is an arbitrary Cauchy constant and $\delta > \delta_0$ as defined in Corollary 6.9.[4]

The constant terms, depending on the initial conditions, are then combined to a single constant C_0 which is $O(1/\epsilon_c)$. (Since ϵ_c is arbitrary, a single Cauchy constant is used for simplicity.) □□

[3] For simplicity, we use the same constant μ to characterize both the mean-absolute and mean-square speed of the unstructured parameter variations.

[4] Notice that, as mentioned in the proof of Corollary 6.9, simpler and perhaps tighter expressions can be used when the parameter derivatives are small uniformly in time.

In other words, the use of a basic estimator with projection and normalization guarantees that the mean-square value of the estimation error is $O(\mu,\ \mu/\gamma)$ with μ representing the average speed of variation of the unstructured part of the plant parameters. It is interesting to observe that higher adaptation gains (γ) tend to decrease but not eliminate the mean-square normalized estimation error while increasing the speed of the parameter estimates, which is $O(\gamma\mu)$ in the mean-square sense. However, what we actually need to make small is the mean-square value of the identification error which can be expressed as

$$e_1 = \epsilon_1 - \hat{\eta}$$

yielding, after some straightforward calculations,

$$\begin{aligned}\left[\int_{t_I}^{t_I+T_I} \frac{e_1^2(t)}{m_2(t)}\,dt\right]^{1/2} \leq\ & C_0 + \left[\int_{t_I}^{t_I+T_I} \frac{\epsilon_1^2(t)}{m_2(t)}\,dt\right]^{1/2} \cdots \\ & + (1+\epsilon_c)\Gamma_1 \left[\int_{t_I}^{t_I+T_I} |\dot{\hat{\theta}}(t)|^2\,dt\right]^{1/2}\end{aligned}$$

with our usual notation. It is now apparent that, for 'successful' identification (in an I/O, mean-square, normalized sense) both $\epsilon_1/\sqrt{m_2}$ and $\dot{\hat{\theta}}$ should be made small in the mean-square sense, something that involves a trade-off in the selection of the adaptation gain. Of course, as $\mu \to 0$ one recovers the standard estimation results for LTI systems (e.g., see [N.A.89, S.B.89]) whereby the value of γ is irrelevant and the normalized identification and estimation errors as well as $\dot{\hat{\theta}}$ are square-integrable.

Also note that the auxiliary filters, the parameter δ_0 and the weighting matrix Q play an essential role in the success of the identification scheme. Qualitative guidelines for their selection can be derived from the above corollary so as to minimize, e.g., the identification error. However, due to the nonlinear interaction between the estimation and control laws, the effect of such parameters on the closed-loop stability/boundedness is quite complicated and cannot be analyzed until the final stage of the analysis.

6.12 Remark: In order for the parameter estimates to converge to their true values in the uniform norm, i.e., $\hat{\theta} \to \hat{\theta}_*$ if $\hat{\theta}_*$ is constant, or $|\hat{\theta}(t) - \hat{\theta}_*(t)|$ to converge to an $O(\mu)$-small residual set for a slowly varying $\hat{\theta}_*(t)$, we need to assume that the regressor vector $w/\sqrt{m_2}$ is persistently exciting ([And.77]). In the special case of slowly TV plants, criteria for persistence of excitation have been derived in [M.G.87]. In general, however, the appearance of the possibly fast TV matrix Π in the regressor vector makes the translation of this condition into a condition on the input signal rather complicated. Although persistence of excitation could be beneficial in the performance of an adaptive controller, it is not necessary for closed-loop boundedness and is not discussed any further in the present study. $\nabla\nabla$

Analogous statements can be made for estimators employing the σ or switching σ-modification. While for the latter the derivations and the resulting expressions are similar to those of Corollary 6.11 and are omitted, the case of the σ-modification is somewhat more interesting. This is partly due to the fact that the a priori knowledge of the set $\mathcal{M}$ is not required and partly due to the involved trade-off's in the selection of both σ and γ. Moreover, from Theorem 6.7 it follows that the various bounds can be expressed in terms of the mean-square value of the derivative of $\hat{\theta}_*$ and therefore only part (b) of Assumption 6.10 is needed.

6.13 Corollary: *Suppose that for an LTV plant satisfying Assumptions 3.1–3.3, $\hat{\theta}_*$ is UB and satisfies Assumption 6.10-b and $U \in L_{\infty,[t_0,t_0+T]}$. Then, the estimator*

$$\dot{\hat{\theta}} = -\gamma\frac{\epsilon_1 w}{m_2} - \sigma(\hat{\theta} - \hat{\theta}_c)$$

guarantees that $\hat{\theta}, \dot{\hat{\theta}}$ are UB on $[t_0, t_0+T]$ and $\hat{\theta}$ converges exponentially fast to the residual set

$$\left\{ \hat{\theta} : |\hat{\theta}(t) - \hat{\theta}_c|_2 \le \sqrt{\frac{\gamma}{4\sigma}} \Gamma_3 \right\}.$$

Furthermore, there exists a constant C_0 depending on the initial conditions but independent of T, t_0, such that for any interval $I = [t_I, t_I + T_I] \subseteq [t_0, t_0 + T]$

1. $\displaystyle\int_{t_I}^{t_I+T_I} \frac{\epsilon_1^2(t)}{m_2(t)}\, dt \le C_0 + \left[\Gamma_1^2 \mu + \frac{\mu}{\gamma\sigma} + K_{\theta*}^2 \frac{\sigma}{\gamma}\right] T_I;$
2. $\displaystyle|\dot{\hat{\theta}}(t)|_2 \le \gamma \Gamma_2 \frac{|\epsilon_1|}{\sqrt{m_2}} + \sqrt{\frac{\gamma\sigma}{2}} \Gamma_3 + \varepsilon(t, t_0);$

where $K_{\theta*}, \Gamma_1, \Gamma_2, \Gamma_3$ are constants depending only on operator gains and the radius of the set $\mathcal{M}$ (their expressions are given in the proof). ▽▽

Proof: The proof follows as a straightforward application of Theorem 6.7 using similar techniques as in Corollary 6.9. Expressions for the estimates of the various bounds can be obtained as

$$K_{\theta*} = ||(|\hat{\theta}_* - \hat{\theta}_c|_2)||_\infty \quad ; \quad \Gamma_3 = g_{2,\delta_0}[G_{\theta_*} - G_{\theta_c}]$$

and Γ_1, Γ_2 are as in Corollaries 6.9 and 6.11. □□

One implication of the last corollary is that the value of σ must be carefully selected for the estimation to be successful. For example, a large ratio γ/σ tends to decrease the mean-square value of $\epsilon_1/\sqrt{m_2}$ while increasing that of $|\dot{\hat{\theta}}|_2$. On the other hand, letting $\sigma = \gamma\sigma'$, the mean-square values of $\epsilon_1/\sqrt{m_2}$ and $|\dot{\hat{\theta}}|_2$ become

$$O\left(\mu, \frac{\mu}{\gamma^2\sigma'}, \sigma'\right) \quad \text{and} \quad O\left(\gamma^2\mu, \frac{\mu}{\sigma'}, \gamma^2\sigma'\right)$$

respectively. Taking $\gamma = 1$ for simplicity, the two can be made small, say less than $\epsilon < 1$, by choosing σ' to be $O(\epsilon)$ and assuming that $\mu = O(\epsilon^2)$. With such a choice of the adaptation parameters σ and γ, however, the parameter estimates can become as large as $O(1/\epsilon)$. This, in turn, gives rise to some technical problems when the results of the corollary are used to establish the boundedness of signals in adaptive control systems. In order to avoid these problems we need to ensure that the identification error can be made small without increasing the bounds of the parameter estimates and the parameter error. One way to achieve this is to restrict the class of parameter variations to those which are slow uniformly in time, as shown by the following corollary.

6.14 Corollary: *Suppose that for an LTV plant satisfying Assumptions 3.1–3.3, $\hat{\theta}_*$ is UB, $||\dot{\hat{\theta}}_*||_\infty \le \mu$ and $U \in L_{\infty,[t_0,t_0+T]}$. Then, the estimator*

$$\dot{\hat{\theta}} = -\gamma \frac{\epsilon_1 w}{m_2} - \sigma(\hat{\theta} - \hat{\theta}_c)$$

guarantees that $\hat{\theta}, \dot{\hat{\theta}}$ are UB on $[t_0, t_0+T]$ and $\hat{\phi}$ converges exponentially fast to the residual set

$$\left\{ \hat{\phi} : |\hat{\phi}(t)|_2 \le K_{\theta*} + \Gamma_1' \sqrt{\frac{\gamma}{2\sigma}\mu + \frac{\mu}{\sigma}} \right\}$$

Furthermore, there exists a constant C_0 depending on the initial conditions but independent of T, t_0, such that for any interval $I = [t_I, t_I + T_I] \subseteq [t_0, t_0 + T]$

1. $\displaystyle\int_{t_I}^{t_I+T_I} \frac{\epsilon_1^2(t)}{m_2(t)}\, dt \le C_0 + \left[\Gamma_1'^2 \mu^2 + \frac{K_{\theta*}^2 \sigma}{\gamma} + \frac{\mu^2}{\gamma\sigma}\right] T_I;$

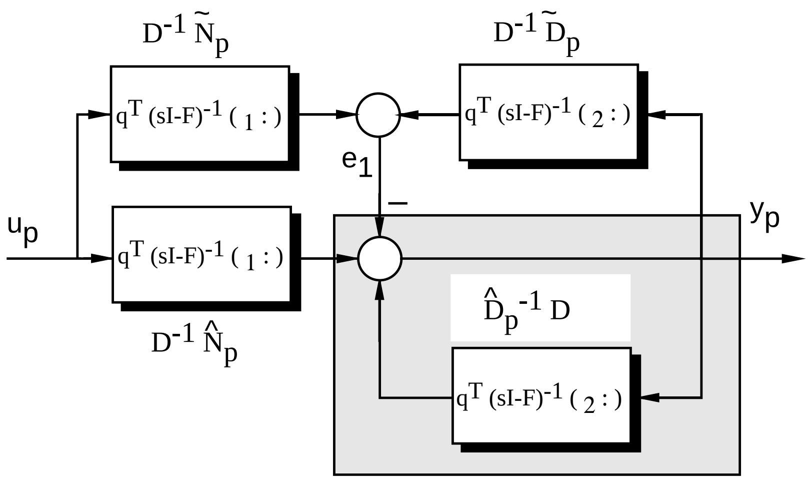

Figure 6.3: Alternative representation of the LTV plant.

2. $|\dot{\hat{\theta}}(t)| \leq \gamma\Gamma_2 \frac{|\epsilon_1|}{\sqrt{m_2}} + 2K_{\theta *}\sigma + \mu + \Gamma_1' \sqrt{\frac{\gamma\sigma}{2}}\mu + \varepsilon(t,t_0)$;

where $K_{\theta *}, \Gamma_2$ *are as in Corollary 6.13 and* Γ_1' *is a constant depending on operator gains (its expression is given in the proof).* ▽▽

Proof: In Appendix VI.

From a different point of view, Corollaries 6.11, 6.13 and 6.14 describe the properties of an identification procedure which identifies a left-fractional representation of an LTV I/O operator within two stable-factor perturbations (see Fig. 6.3). These perturbations have the property that their combined output is 'small' in a mean-square, normalized sense. This can be seen by expressing y_p in terms of the identification error as

$$\begin{aligned} y_p &= G(s)[u_p\theta_1] + G(s)[y_p\theta_2] - e_1 + \varepsilon \\ e_1 &= G(s)[u_p\phi_1] + G(s)[y_p\phi_2] = \epsilon_1 - \hat{\eta}. \end{aligned}$$

Rewriting the above equation in terms of operators we have

$$\begin{aligned} y_p = & \; D^{-1}(s)\hat{N}_p(s,t)[u_p] + D^{-1}(s)\{D(s) - \hat{D}_p(s,t)\}[y_p] \\ & -D^{-1}(s)\tilde{N}_p(s,t)[u_p] - D^{-1}(s)\tilde{D}_p(s,t)[y_p] \end{aligned}$$

where the coefficients of $\hat{D}_p(s,t), \hat{N}_p(s,t)$ and $\tilde{D}_p(s,t), \tilde{N}_p(s,t)$ are easily obtained from θ, ϕ and $D(s) = \det(sI-F)$.[5] In other words the unknown plant $D_p^{-1}(s,t)N_p(s,t)$ is effectively replaced by its estimate with the known I/O operator $\hat{D}_p^{-1}(s,t)\hat{N}_p(s,t)$ and the stable factor perturbations $D^{-1}(s)\tilde{N}_p(s,t)$ and $D^{-1}(s)\tilde{D}_p(s,t)$. The benefits are quite apparent. Since θ is known, a controller can now be designed for the estimated I/O operator of the plant $\hat{D}_p^{-1}(s,t)\hat{N}_p(s,t)$. Consequently, the closed-loop stability properties are determined by the robustness properties of the controller with respect to the stable factor perturbations e_1 which are not necessarily of small gain but are small in a mean-squared, normalized sense. Such a design is studied in more detail in a forthcoming chapter. At this point, however, we should emphasize the importance of the auxiliary filters used in the estimator. Not only do they determine the properties of the perturbation e_1 but they also affect the sensitivity operators $e_1 \mapsto y_p$ and $e_1 \mapsto u_p$ which are the key operators involved in the analysis of the properties of adaptive controllers.

[5]Note that $\tilde{D}_p(s,t)$ is a PDO of degree $n-1$.

6.15 Example: To demonstrate the advantages of modeling the plant as in Lemmas 3.10 or 3.11 and using the structured-parameter-variations approach, let us consider the case where the plant I/O description is given by

$$s^2[y_p] + s[a_1 y_p] + a_2 y_p = s[u_p] + b_1 u_p \tag{6.14}$$

where

$$a_1 = 20 + 12\sin 2t \ ; \ a_2 = 6\cos 2t \ ; \ b_1 = -1 \tag{6.15}$$

Let us also suppose that we know a priori that the plant parameters are of the form

$$a_1 = c_1 + c_2 \sin 2t \ ; \ a_2 = c_3 + c_4 \cos 2t \ ; \ b_1 = c_5 \tag{6.16}$$

where $N_p(s,t)$ is known to be a monic PDO of degree one and $c_1 - c_5$ are some unknown constants whose range, say $c_{i,min} \le c_i \le c_{i,max}$, is known a priori.

Following the guidelines of the presented analysis let us consider the second order auxiliary filter:

$$G(s) = q^\top (sI - F)^{-1}, \quad q^\top = [1,0] \ ; \ F = \begin{pmatrix} -5 & 1 \\ -6 & 0 \end{pmatrix} \tag{6.17}$$

from which $D(s) = s^2 + 5s + 6$. Applying Lemma 3.10, we obtain the following parametrization for the plant

$$y_p = G(s)[u_p \theta_{1*}] + G(s)[y_p \theta_{2*}]$$

where

$$\theta_{1*} = \begin{bmatrix} 1 \\ \hat{\theta}_{1*} \end{bmatrix} \ ; \ \theta_{2*} = \begin{bmatrix} \hat{\theta}_{2*} + \hat{\theta}_{4*} \sin 2t \\ \hat{\theta}_{3*} + \hat{\theta}_{5*} \cos 2t \end{bmatrix}$$

and $\hat{\theta}_*^\top = [-1, -15, 6, -12, -6]$ is an unknown constant vector, to be estimated. Next, employing the notion of structured parameter variations and since the leading coefficient of $N_p(s,t)$ is known, we may express y_p as

$$y_p = sD^{-1}(s)[u_p] + G(s)[u_p \Pi_1]\hat{\theta}_* + G(s)[y_p \Pi_2]\hat{\theta}_*$$

where,

$$\Pi_1(t) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \ ; \ \Pi_2(t) = \begin{pmatrix} 0 & 1 & 0 & \sin 2t & 0 \\ 0 & 0 & 1 & 0 & \cos 2t \end{pmatrix}$$

Thus, the estimation error is constructed as

$$\epsilon_1 = w^\top \hat{\theta} - y_p + sD^{-1}(s)[u_p]$$

where $w^\top = [(0,1)w_1,\ w_2^\top,\ (1,0)w_3,\ (0,1)w_4]$,

$$\dot{w}_1 = F^\top w_1 + q u_p \quad ; \quad \dot{w}_2 = F^\top w_2 + q y_p$$
$$\dot{w}_3 = F^\top w_3 + q y_p \sin 2t \quad ; \quad \dot{w}_4 = F^\top w_4 + q y_p \cos 2t$$

and $\hat{\theta}$ being the estimate of $\hat{\theta}_*$. Since the range of each element of $\hat{\theta}_*$ can be directly deduced from the range of the c_i's, we may use the following adaptive law to update $\hat{\theta}$

$$\dot{\hat{\theta}} = \mathcal{P}\left(-\gamma \frac{\epsilon_1 w}{m_2}\right) \tag{6.18}$$

with $\mathcal{P}$ as given by (6.5) and $\delta_0 < 2$. Finally, the estimate θ_i of θ_{i*} is

$$\theta_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \Pi_1 \hat{\theta} \ ; \ \theta_2 = \Pi_2 \hat{\theta}$$

from which the estimates of the plant parameters, denoted by $(\hat{\cdot})$, are given by

$$\hat{b}_1 = (0,1)\theta_1 \; ; \; \hat{a}_1 = -(1,0)\theta_2 + 5 \; ; \; \hat{a}_2 = -(0,1)\theta_2 + 6 \tag{6.19}$$

Thus, according to Corollary 6.11, the estimator (6.18) guarantees that $\epsilon_1/\sqrt{m_2}$ and $\dot{\hat{\theta}}$ is square integrable, for as long as $U \in L_{\infty,[t_0,t_0+T]}$. Furthermore, if $u_p/\sqrt{m_2}$ is UB, it also follows that $\epsilon_1/\sqrt{m_2}$ is uniformly continuous and therefore converge to zero as $t \to \infty$, provided of course that $u_p \in L_\infty^e$. However, whether this condition is met or not depends also on the way u_p is generated (for further discussion, see Chapter 8).

It should be noted that in the case there is some ambiguity in the frequency of the $\sin w_0 t$ terms, e.g., the true frequency is $w_0 + \delta$, we can decompose the sine terms as

$$\sin(w_0 + \delta)t = \sin w_0 t \cos \delta t + \cos w_0 t \sin \delta t$$

The last equation suggests that an additional parameter should be used to estimate the $\cos w_0 t$ terms with the corresponding elements of $\hat{\theta}_*$s being slowly TV for small δ (similarly for the cosine terms). In this case, Lemma 3.10 can still be used to establish the smallness in the mean-square of the normalized estimation error even if w_0 is large. ▽▽

6.3.2 Non-Smooth Parameter Variations

For our next topic of study we consider the identification problem for plants with non-smooth parameter variations. In this case, the plant may not be described with a convenient PDO/PIO factorization (even with non-smooth coefficients) which would allow the straightforward application of our previous results. Instead, we rely on Lemma 3.11 to parametrize an LTV plant of known order, with a state-space representation satisfying Assumptions 3.4–3.6. This lemma together with Corollary 2.60 imply that the output of the LTV plant at time t can be expressed as

$$\begin{aligned} y_p &= w^\top \hat{\theta}_* + \eta_s + \tilde{\eta} + \eta_J + q^\top \Phi_F(t,t_0) x_F(t_0) \\ \eta_s &= -G(s)\{G'_\pi[QU]\dot{\hat{\theta}}_*^s\} \\ \tilde{\eta} &= G(s)[\tilde{A}_F x_F + \tilde{b}_F u_p]\} + \tilde{c}_F^\top x_F \\ \eta_J &= \sum_{t_j \le t} q^\top \Phi_F(t,t_j) \left\{ [\bar{P}(t_j) - I] x_F(t_j^-) + G'_\pi[QU](t_j)\hat{\theta}_{*j}^J \right\} \end{aligned} \tag{6.20}$$

where $\hat{\theta}_*^s$ is the Lipschitz continuous part and $\hat{\theta}_*^J = \sum_{t_j \le t} \hat{\theta}_{*j}^J$ is the jump part of $\hat{\theta}_*$ and w is the usual regressor vector as in the smooth parameter case; the rest of the notation is as in Lemma 3.11 Although such a parametrization is quite more complicated than before, it nevertheless retains the same basic properties required for an application of Theorems 6.6 and 6.7. The only difference here is that the 'noise' signal η is composed of three terms:

1. the term η_s, which depends on the speed of variation of the Lipschitz part of $\hat{\theta}_*$ and is characterized by the parameter μ;

2. the term $\tilde{\eta}$ which incorporates the effects of smooth approximations of the plant parameters and possible short excursions of the plant parameters to regions where strong controllability or observability fail; this term is characterized by the parameter μ' which is typically expected to be very small ($\ll 1$);

3. the term η_J describing the cumulative effects of the jump part of the plant parameters and characterized by the usually very small parameter ν.

Thus, non-smooth-parameter analogs of Corollaries 6.11 and 6.13, characterizing the I/O quality of the identification by means of the three parameters μ, μ', ν, can be derived in a straightforward manner from the general theorems of Section 6.2. In the following statements, we consider an LTV plant satisfying Assumptions 3.4–3.6 and the corresponding parameter vector θ_*, as defined in Lemma 3.11. Without loss of generality, we assume that

$$\theta_* = \Pi \hat{\theta}_*$$

where Π is a piecewise smooth, UB matrix and $\hat{\theta}_*$ is a piecewise smooth vector with possible discontinuities at $t = t_j$, $j = 1, 2, \ldots$.

6.16 Corollary: *Under these conditions, suppose that $\hat{\theta}_*$ and $\hat{\theta}_*^s$ satisfy Assumptions 6.3 and 6.10, respectively. Further suppose that $U \in L_{\infty,[t_0,t_0+T]}$ and consider the estimator*

$$\dot{\hat{\theta}} = \mathcal{P}\left(-\gamma \frac{\epsilon_1 w}{m_2}\right) \ ; \ \ \hat{\theta}(t_0) \in \mathcal{M}$$

Then, there exist $\nu_0, \mu_0' > 0$, depending on the value of δ_0 used in m_2, such that for any $\nu \in [0, \nu_0)$, $\mu' \in [0, \mu_0')$, $\hat{\theta}, \dot{\hat{\theta}}$ are UB on $[t_0, t_0 + T]$ ($|\hat{\theta} - \hat{\theta}_c| \leq M_0 + \epsilon_*$); furthermore, there exist constants $C_0, \tilde{K}, K_J$ such that for any interval $I = [t_I, t_I + T_I] \subseteq [t_0, t_0 + T]$

1. $\displaystyle\int_{t_I}^{t_I+T_I} \frac{\epsilon_1^2(t)}{m_2(t)}\, dt \leq C_0 + \left[\Gamma_1^2 + \frac{K_\phi}{\gamma}\right] \mu T_I + \tilde{K}\mu' T_I + K_J^2 \left(1 + \frac{1}{\gamma}\right) \nu T_I;$
2. $\displaystyle|\dot{\hat{\theta}}(t)| \leq \gamma \Gamma_2 \frac{|\epsilon_1|}{\sqrt{m_2}} + \varepsilon(t, t_0);$

where $K_\phi, \Gamma_1, \Gamma_2$ are as in Corollary 6.11. The constants $\tilde{K}, K_J$ depend on the perturbation part of the plant and the size of jumps in the nominal plant parameters and their derivatives, respectively but they are independent of initial conditions whose effect is incorporated in C_0; all three constants are independent of T, t_0. ▽▽

Proof: In Appendix VI.

6.17 Corollary: *Under the conditions stated above, suppose that $\hat{\theta}_*$ is UB and $\hat{\theta}_*^s$ satisfies Assumption 6.10-b. Further suppose that $U \in L_{\infty,[t_0,t_0+T]}$ and consider the estimator*

$$\dot{\hat{\theta}} = -\gamma \frac{\epsilon_1 w}{m_2} - \sigma(\hat{\theta} - \hat{\theta}_c)$$

Then, there exist $\nu_0, \mu_0' > 0$, depending on the value of δ_0 used in m_2, such that for any $\nu \in [0, \nu_0)$, $\mu' \in [0, \mu_0')$, $\hat{\theta}, \dot{\hat{\theta}}$ are UB on $[t_0, t_0 + T]$ and $\hat{\theta}$ converges exponentially fast to the residual set

$$\left\{\hat{\theta} : |\hat{\theta}(t) - \hat{\theta}_c|_2 \leq \sqrt{\frac{\gamma}{4\sigma}\Gamma_3'}\right\}$$

where Γ_3' is a constant, approaching $\Gamma_3 = g_{2,\delta_0}[G_{\theta_*} - G_{\theta_c}]$ as the perturbation part of the plant and the jumps in the plant parameters and their derivatives vanish. Furthermore, there exist constants $C_0, \tilde{K}, K_J$ such that for any interval $I = [t_I, t_I + T_I] \subseteq [t_0, t_0 + T]$

1. $$\int_{t_I}^{t_I+T_I} \frac{\epsilon_1^2(t)}{m_2(t)}\, dt \leq C_0 + \left[\Gamma_1^2 \mu + \frac{\mu}{\gamma\sigma} + K_{\theta*}^2 \frac{\sigma}{\gamma}\right] T_I + \tilde{K}\mu' T_I + K_J^2 \left(1 + \frac{1}{\gamma}\right) \nu T_I;$$

2. $|\dot{\hat{\theta}}(t)|_2 \leq \gamma\Gamma_2 \frac{|\epsilon_1|}{\sqrt{m_2}} + \sqrt{\frac{\gamma\sigma}{2}}\Gamma_3' + \varepsilon(t, t_0)$;

where $K_{\theta*}$, Γ_1, Γ_2, Γ_3 are as in Corollaries 6.11 and 6.13. The constants $\tilde{K}, K_J$ depend on the perturbation part of the plant and the size of jumps in the nominal plant parameters and their derivatives, respectively but they are independent of initial conditions whose effect is incorporated in C_0; all three constants are independent of T, t_0. ▽▽

Proof: As in Corollary 6.16, but using the expressions of Corollary 6.13 rather than 6.11. □□

Thus, Corollaries 6.16 and 6.17 describe the properties of our basic estimation algorithms when used to identify an LTV plant with non-smooth parameter variations. As in the smooth parameter case, the quality of identification —in an I/O, normalized sense— depends on the average speed of variations of the smooth part of $\hat{\theta}_*$ while the effect of the discontinuities and perturbations in the state-space description of the LTV plant introduces two additional terms $O(\nu)$ and $O(\mu')$ in the various expressions.

In this case, however, μ' and ν should be sufficiently small. This requirement arises from the need to guarantee the uniform boundedness of the normalized signals, including the plant state vector, in order to obtain expressions which are uniform with respect to the interval of integration and the initial time. Notice that large μ' or ν may result in the overall plant not being uniformly observable[6] and a consequent failure of $x_F/\sqrt{m_2}$ to be UB. If, on the other hand, we assume that, in addition to Assumptions 3.4–3.6, the LTV plant (nominal and perturbation part) is uniformly observable, then no restriction on μ' and ν is necessary.

Finally, as mentioned in the previous subsection, when an estimator employing the σ-modification is used for adaptive control purposes, we need to impose some additional uniformity conditions on the size of the perturbation terms. More precisely, we assume that

$$||\dot{\hat{\theta}}_*^s||_\infty \leq \mu \quad ; \quad |\tilde{A}|, |\tilde{b}|, |\tilde{c}| \leq \mu' \quad ; \quad t_{j+1} - t_j \geq \frac{1}{\nu}$$

uniformly in t and j, where as usual the superscript 's' denotes the smooth part of the parameters, '$\tilde{\cdot}$' refers to the state-space perturbations due to smooth approximations and t_j are the discontinuity points.

6.18 Corollary: *Under the Assumptions of Corollary 6.17 as restricted by the above conditions, for any $\sigma > 0$ there exist $\nu_0, \mu_0' > 0$ such that for any $\nu \in [0, \nu_0)$ and $\mu' \in [0, \mu_0')$, $\hat{\theta}, \dot{\hat{\theta}}$ are UB on $[t_0, t_0 + T]$ and $\hat{\phi}$ converges exponentially fast to a residual set satisfying*

$$|\hat{\phi}(t)|_2 \leq O\left(K_{\theta*}, \frac{\mu}{\sigma}, \mu\sqrt{\frac{\gamma}{\sigma}}, \mu'\sqrt{\frac{\gamma}{\sigma}}\right) + K_J O\left(1, \sqrt{\frac{\gamma}{\sigma}}\right) e^{-\beta(t-t_j)} \ ; \quad t \geq t_j$$

where $K_{\theta*}$ is a constant as in Corollary 6.13, K_J is a constant depending on the size of the jumps in the nominal plant parameters and their derivatives and for any $\sigma > 0$, $\beta(\sigma) > 0$ is a constant.

Furthermore, there exists a constant C_0 depending on the initial conditions but independent of T, t_0, such that for any interval $I = [t_I, t_I + T_I] \subseteq [t_0, t_0 + T]$

1. $\int_{t_I}^{t_I+T_I} \frac{\epsilon_1^2(t)}{m_2(t)} dt \leq C_0 + O\left(\mu^2, \mu'^2, K_J^2\nu, \frac{K_{\theta*}^2\sigma}{\gamma}, \frac{\mu^2}{\gamma\sigma}, \frac{K_J^2\nu}{\gamma}\right) T_I$;

2. $|\dot{\hat{\theta}}(t)| \leq O(\gamma) \frac{|\epsilon_1|}{\sqrt{m_2}} + O\left(K_{\theta*}\sigma, \mu, \mu\sqrt{\gamma\sigma}, \mu'\sqrt{\gamma\sigma}\right) \ldots$

$$+ K_J O\left(\sigma, \sqrt{\gamma\sigma}\right) e^{-\beta(t-t_j)} + \varepsilon(t, t_0) \ ; \quad t \geq t_j;$$

▽▽

[6] This is because Assumptions 3.5 and 3.6 are on the nominal part of the plant only.

Proof: In Appendix VI.

Note that due to the complexity and increased conservatism of the expressions, we state Corollary 6.18 using the simplified notation $O(x_1, x_2, \ldots) \leq K_1 x_1 + K_2 x_2 + \cdots$ where K_i are constants. All such constants are independent of the parameter estimates. We do, however, keep track of the important adaptation parameters γ and σ and the radius of the parametric uncertainty set $K_{\theta*}$ for later discussion.

6.19 Remark: Although, for estimation purposes, it is possible to allow discontinuities in $\Pi(t)$ at instants other than at t_j, this increases the total number of discontinuity points that should be accounted for, when the estimator is used in an adaptive control context. For this reason, and in order to maintain some uniformity in the notation, we assume that the entries of $\Pi(t)$ are smooth inside each interval (t_j, t_{j+1}). $\nabla\nabla$

APPENDIX VI

Proof of Theorem 6.6:

With the properties of $\mathcal{P}$, y, w and η, the vectorfield in (6.4) satisfies the Caratheodory assumptions, ensuring the local existence of an absolutely continuous function θ satisfying (6.4) almost everywhere. Furthermore, this solution is unique (see [C.L.55], pp. 48–51) and can be extended in the whole interval $[t_0, t_0 + T]$. Also, from the definition of $\mathcal{P}$ and with $\theta(t_0) \in \mathcal{M}$, the distance of $\theta(t)$ from the set $\mathcal{M}$ is at most ϵ_* (see Examples 6.4 and 6.5 for the definition of $\mathcal{P}$ and ϵ_* for sets $\mathcal{M}$ defined in an L_∞ or L_2 sense). Hence, the first part of the theorem follows.

For the second part, let t_i, $i = 0, 1, \ldots, n_T$, $t_i \leq t_{i+1} \leq t_0 + T$, denote the points of discontinuity of θ_* in the interval $[t_0, t_0 + T]$. Further, define the parameter error $\phi = \theta - \theta_*$. It follows that ϕ satisfies the differential equation

$$\dot{\phi} = \mathcal{P}(-\gamma \epsilon_1 w) - \dot{\theta}_*^s \; ; \quad t \in [t_i, t_{i+1})$$

with initial conditions

$$\phi(t_0) = \theta(t_0) - \theta_*(t_0)$$

and boundary conditions at each discontinuity point

$$\phi(t_i) = \phi(t_i^-) - [\theta_*(t_i^+) - \theta_*(t_i^-)]$$

where t_i^-, t_i^+ are used to denote left and right limits respectively.

Next, consider the positive definite function $V = \frac{1}{2\gamma}\phi^\top \phi$. The derivative of V along the trajectories of (6.4) is given by

$$\dot{V} = -\phi^\top \mathcal{P}(\epsilon_1 w) - \frac{1}{\gamma}\phi^\top \dot{\theta}_*^s \; ; \quad t \in [t_i, t_{i+1})$$

From the definitions of $\mathcal{P}$ and $\mathcal{M}$ and after some simple geometry, it follows that when the projection is active ($\neq 1$), V decreases faster than the V corresponding to a vectorfield without projection. Hence,

$$\dot{V} \leq -\epsilon_1 \phi^\top w - \frac{1}{\gamma}\phi^\top \dot{\theta}_*^s \; ; \quad t \in [t_i, t_{i+1})$$

from which, using (6.3) and taking norms of the sign-indefinite terms we obtain

$$\dot{V} \leq -\epsilon_1^2 + |\epsilon_1 \eta| + \frac{1}{\gamma}\|\phi\|_\infty |\dot{\theta}_*^s|_i \; ; \quad t \in [t_i, t_{i+1})$$

where the subscript i denotes the functional norm induced by the norm of ϕ (typically the norm used to define $\mathcal{M}$). Moreover, V being absolutely continuous in $[t_i, t_{i+1})$, a simple completion of squares and integration yields

$$\int_{t_i}^{t} \epsilon_1^2(\tau)\, d\tau \leq 2[V(t_i^+) - V(t)] + \int_{t_i}^{t} \eta^2(\tau)\, d\tau + \frac{K_\phi}{\gamma} \int_{t_i}^{t} |\dot{\theta}_*^s(\tau)|_i\, d\tau \,;\; t \in [t_i, t_{i+1})$$

from which, V being UB, property (1) follows. Notice that expressing $\dot{V}$ in terms of $w^\top \phi$ rather than ϵ_1, the same inequality holds for $(w^\top \phi)^2$. Finally, property (2) follows immediately from (6.4) and Assumption 6.2. □□

Proof of Theorem 6.7:

While the existence and uniqueness of an absolutely continuous θ satisfying (6.7) follow as in Theorem 6.6, boundedness can be shown by considering the positive definite function $V_1 = \frac{1}{2}(\theta - \theta_c)^\top(\theta - \theta_c)$. Then, along the trajectories of (6.7)

$$\dot{V}_1 \leq \frac{\gamma}{4}(y - w^\top \theta_c)^2 - 2\sigma V_1$$

from which, θ is uniformly ultimately bounded with an upper bound as given in the theorem. Further, a bound on $\dot{\theta}$ is simply derived by taking norms on both sides of (6.7). It is important to notice that, modulo exponentially decaying terms, these bounds are independent of the initial conditions $\theta(t_0)$.

For the rest of the properties, let $V = \frac{1}{2\gamma}\phi^\top \phi$ where $\phi = \theta - \theta_*$. Then, as in Theorem 6.6

$$\dot{V} \leq -\epsilon_1^2 + |\epsilon_1 \eta| - \frac{\sigma}{\gamma}|\phi|^2 + \frac{\sigma}{\gamma}|\phi||\theta_* - \theta_c + \frac{1}{\sigma}\dot{\theta}_*^s| \;;\quad t \in [t_i, t_{i+1})$$

while, from the boundedness of θ and θ_*, V is UB. The inequalities of the theorem are now easily obtained after completing the squares of the above inequality and integrating both sides. □□

Proof of Corollary 6.9:

The boundedness of $\hat{\theta}$ and its derivative are obtained as a straightforward application of Theorem 6.6 with the affine model (6.13), which also implies that

$$\begin{aligned}\int_{t_I}^{t_I+T_I} \left(\frac{\epsilon_1(t)}{m_p^{1/p}(t)}\right)^2 dt \quad &\leq \quad C_0 + K_\phi \int_{t_I}^{t_I+T_I} |\dot{\hat{\theta}}_*(t)|_i\, dt \\ &\quad + \int_{t_I}^{t_I+T_I} \left(\frac{\eta(t) + \varepsilon(t, t_0)}{m_p^{1/p}(t)}\right)^2 dt\end{aligned}$$

where C_0 is a constant $O(M_0^2/\gamma)$, M_0 being the radius of the set $\mathcal{M}$ and K_ϕ is a constant due to pulling $|\hat{\phi}(t)|$ outside the integral. When the projection $\mathcal{P}$ is given by (6.5), $||\theta - \theta_c||_\infty \leq M_0 + \epsilon_*$ which implies that $K_\phi \leq 4M_0 + 2\epsilon_*$, while the induced norm of $\dot{\hat{\theta}}_*(t)$ is the l_1 vector norm. Similar relations are also obtained if the set $\mathcal{M}$ and the projection $\mathcal{P}$ are defined through a Euclidean norm.

Further, for the first property of the corollary, we notice that $\eta/m_p^{1/p}$ is UB and

$$m_p(t) \geq \exp[-p\delta_0(t - \tau)]m_p(\tau) \;, \quad \forall\, t_0 \leq \tau \leq t \;. \tag{6.21}$$

Hence, we can express $\eta(t)$, $t \in [t_I, t_I + T_I]$ as

$$\eta(t) = (G(s)\{G'_\pi[QU]\dot{\hat{\theta}}_*\})(t) + \beta(t_I)e^{-\delta(t - t_I)}$$

where the functions $U, \dot{\hat{\theta}}_*$ are taken as zero for $t < t_I$. This, of course, introduces an effective initial condition term $\beta(t_I)$ for which we have that $\beta(t_I)/m_p^{1/p}(t_I)$ is UB. We may now invoke Lemma 2.56, with operator gains induced by the norms of the $L_p(\delta)$ as well as the underlying $\mathbf{R}^n$ spaces, to write

$$\begin{aligned}
\frac{|\eta(t)|}{m_p^{1/p}(t)} &\leq g_{p,\delta}[G]\frac{\mathcal{E}_{-\delta}(t)\|(G'_\pi[QU]\dot{\hat{\theta}}_*)_t\|_{p,\delta}}{m_p^{1/p}(t)} + \frac{|\beta(t_I)|}{m_p^{1/p}(t_I)}e^{-(\delta-\delta_0)(t-t_I)} \\
&\leq g_{p,\delta}[G]\left[\int_{t_I}^t e^{-p(\delta-\delta_0)(t-\tau)}\frac{|G'_\pi[QU]|^p(\tau)}{m_p(\tau)}|\dot{\hat{\theta}}_*(\tau)|^p\, d\tau\right]^{1/p} \ldots \\
&\quad + \frac{|\beta(t_I)|}{m_p^{1/p}(t_I)}e^{-(\delta-\delta_0)(t-t_I)} \\
&\leq g_{p,\delta}[G]g_{p,\delta_0}[G'_\pi]\left[\int_{t_I}^t e^{-p(\delta-\delta_0)(t-\tau)}|\dot{\hat{\theta}}_*(\tau)|^p\, d\tau\right]^{1/p} \ldots \\
&\quad + \frac{|\beta(t_I)|}{m_p^{1/p}(t_I)}e^{-(\delta-\delta_0)(t-t_I)}
\end{aligned}$$

from which, property (1) follows as an immediate application of the Schwarz inequality.

Note that, alternatively, one may also derive a bound for $|\eta|$ in terms of the $g_{1,\delta}$-gain of G, the γ_{2,δ_0}-gain of G_π and the conjugate-index $L_q(\delta)$-norm of $\dot{\hat{\theta}}_*$. Omitting for simplicity the exponentially decaying terms, this procedure is outlined below.

$$\begin{aligned}
|\eta(t)| &\leq g_{1,\delta}[G]\mathcal{E}_{-\delta}\|(G'_\pi[QU]\dot{\hat{\theta}}_*)_t\|_{1,\delta} \\
&\leq g_{1,\delta}[G]\mathcal{E}_{-\delta}\|(G'_\pi[QU])_t\|_{p,\delta_0}\|(\dot{\hat{\theta}}_*)_t\|_{q,\delta}\ ; \quad \begin{cases} \delta > \delta_0 \\ p^{-1}+q^{-1}=1 \end{cases} \\
&\leq g_{1,\delta}[G]\gamma_{p,\delta_0}[G'_\pi]m_p^{1/p}(t)\mathcal{E}_{-\delta+\delta_0}\|(\dot{\hat{\theta}}_*)_t\|_{q,\delta-\delta_0}
\end{aligned}$$

More efficient bounds can be derived in the special case of TV parameters whose derivatives are small, uniformly in time. Assuming, for example, that $|\dot{\hat{\theta}}_*| \leq \mu$ and following the same procedure we get

$$\begin{aligned}
\frac{|\eta(t)|}{m_p^{1/p}(t)} &\leq g_{p,\delta_0}[G]\frac{\mathcal{E}_{-\delta_0}(t)\|(G'_\pi[QU]\dot{\hat{\theta}}_*)_t\|_{p,\delta_0}}{m_p^{1/p}(t)} + \frac{|\beta(t_I)|}{m_p^{1/p}(t_I)}e^{-\delta_0(t-t_I)} \\
&\leq g_{p,\delta_0}[G]\gamma_{p,\delta_0}[G'_\pi]\mu + \frac{|\beta(t_I)|}{m_p^{1/p}(t_I)}e^{-\delta_0(t-t_I)}
\end{aligned}$$

Next, property (2) follows from Theorem 6.6 and using Lemma 2.56 to write $|w|/m_p^{1/p} \leq g_{p,\delta_0}[G_\pi] + \beta' \exp[-\delta(t-t_0)]$. The constant C''_0 is then used to absorb the integral of the exponentially decaying term (notice that $\epsilon_1/m_p^{1/p}$ is UB since $\hat{\theta}$ is UB).

Finally, it is straightforward to extend the proof to admit non-zero initial conditions in the auxiliary filters. In this case

$$\epsilon_1 = w^\top \hat{\phi} + \eta + \varepsilon + \varepsilon' \hat{\theta}$$

where the last term is due to the initial conditions in the auxiliary filters and the rest of the quantities are as before. The same arguments can now be used to show the boundedness of $\hat{\theta}$ from which the contribution of $\varepsilon'\hat{\theta}$ can be included in the C_0 constants. □□

Proof of Corollary 6.14:

While the existence and uniqueness of an absolutely continuous $\hat{\theta}$ satisfying the estimator ODE follow as in Theorem 6.6, boundedness can be shown by considering the positive definite function $V = \frac{1}{2}\hat{\phi}^\top \hat{\phi}$. Then,

$$\dot{V} = -\gamma\frac{\epsilon_1^2 - \epsilon_1\eta}{m_2} - \sigma\|\hat{\phi}\|^2 + (\sigma\hat{\phi}_c - \dot{\hat{\theta}}_*)^\top \hat{\phi} \tag{6.22}$$

where $\hat{\phi}_c = \hat{\theta}_c - \hat{\theta}_*$. Completing the squares in (6.22) we obtain

$$\dot{V} \le -\sigma V + \frac{\gamma}{4}\frac{|\eta|^2}{m_2} + \frac{1}{2\sigma}(\sigma K_{\theta*} + \mu)^2 \tag{6.23}$$

where $K_{\theta*}$ is as in Corollary 6.13. Further, as in Corollary 6.9, $\eta^2/m_2 \le \Gamma_1'^2\mu^2 + \varepsilon$ where $\Gamma_1' = (1+\epsilon_c)g_{2,\delta_0}[G]\gamma_{2,\delta_0}[G'_\pi]$, ϵ_c is a Cauchy constant and ε is an exponentially decaying term due to initial conditions. Substituting this expression in (6.23), the first part of the corollary follows.

For the rest of the properties, we use again (6.22) and complete the squares to obtain

$$2\dot{V} \le -\gamma\frac{\epsilon_1^2 - \eta^2}{m_2} + \frac{1}{2\sigma}(\sigma K_{\theta*} + \mu)^2$$

Integrating both sides of the last inequality we obtain property (1) while property (2) follows by taking the norms of both sides of the estimator ODE. □□

Proof of Corollary 6.16:

From (6.20), y_p has the same general form as in the smooth parameter case and therefore, in order to apply the results of Theorem 6.6 and Corollary 6.11, it suffices to show that $\eta_s, \tilde{\eta}$ and η_J, normalized by $\sqrt{m_2}$, are UB and have similar average properties as η. Indeed, $\eta_s/\sqrt{m_2}$ can be handled the same way as $\eta/\sqrt{m_2}$ in the smooth parameter case, with θ_*^s replacing θ_* in the various expressions. On the other hand, for $\tilde{\eta}$ we have that

$$|G(s)[\tilde{A}_F x_F](t)| \le g_{2,\delta}[G]\left\{\int_{t_0}^t e^{-2\delta(t-\tau)}|\tilde{A}_F(\tau)|^2|x_F(\tau)|^2\,d\tau\right\}^{1/2}$$

Since for μ', ν sufficiently small, x_F^2/m_2 is UB for as long as $U \in L_{\infty,[t_0,t_0+T]}$ (see Lemma 3.11) and using the fact

$$m_2(t) \ge e^{-2\delta_0(t-\tau)}m_2(\tau)\ ;\quad t \ge \tau$$

we have that for all $\mu' \in [0, \mu_0')$, $\nu \in [0, \nu_0)$

$$\frac{|G(s)[\tilde{A}_F x_F](t)|^2}{m_2} \le g_{2,\delta}^2[G]K_F\int_{t_0}^t e^{-2(\delta-\delta_0)(t-\tau)}|\tilde{A}_F(\tau)|^2\,d\tau + \varepsilon(t,t_0)$$

for some constant K_F, independent of the initial conditions which are included in the exponentially decaying term $\varepsilon(t,t_0)$. Next,

$$|G(s)[\tilde{b}_F u_p](t)| \le g_{1,\delta}[G]\int_{t_0}^t e^{-\delta(t-\tau)}|\tilde{b}_F(\tau)||u_p(\tau)|\,d\tau$$

and therefore, using the Schwarz inequality, we obtain

$$\frac{|G(s)[\tilde{b}_F u_p](t)|^2}{m_2(t)} \le g_{1,\delta}^2[G]\int_{t_0}^t e^{-2(\delta-\delta_0)(t-\tau)}|\tilde{b}_F(\tau)|^2\,d\tau \ldots \frac{\int_{t_0}^t e^{-2\delta_0(t-\tau)}|u_p(\tau)|^2\,d\tau}{m_2}$$

the last factor being UB. Finally,

$$|\tilde{c}_F^\top(t)x_F(t)|^2/m_2 \le K_F|\tilde{c}_F(t)|^2 + \varepsilon(t,t_0).$$

Accumulating all the terms it follows that $\tilde{\eta}^2(t)/m_2(t)$ is UB and, using Assumption 3.4, we get that in an interval $I \subseteq [t_0, t_0+T]$

$$\int_{t_I}^{t_I+T_I}\frac{\tilde{\eta}^2(t)}{m_2(t)}\,dt \le C_0 + \tilde{K}'\mu' T_I$$

for some constant $\tilde{K}'$ independent of the initial conditions.

Further, a similar property can be established for the term η_J under Assumption 3.6. For this, we make use of the following intermediate result

- Let t_j, $j = 1, 2, \ldots$ be an increasing sequence and for $t, T \geq 0$ define k, l such that $t_k \leq t < t + T \leq t_m$. Suppose that there exist constants $C, T > 0$ such that $l - k \leq C$ $\forall t \geq 0$. Then, for any $\alpha > 0$, there exists a constant C' such that

$$\sum_{j=1}^{N} e^{-\alpha(t_N - t_j)} \leq C' \; ; \quad \forall N > 0$$

Proof: Consider the intervals $[t_N - (\lambda+1)T, t_N - \lambda T]$, $\lambda = 0, 1, 2 \ldots, \Lambda$; $\Lambda : t_N - \Lambda T \leq 0$. Then each one of these intervals contains at most C terms of the sequence and therefore

$$\sum_{j=1}^{N} e^{-\alpha(t_N - t_j)} \leq C \sum_{\lambda=0}^{\Lambda} e^{-\alpha\lambda T} \leq C' = C \frac{1}{1 - e^{-\alpha T}} \; ; \quad \forall N > 0$$

□□

In view of the above statement, we may now use the properties of the normalizing signal and the fact that $x_F/\sqrt{m_2}$ is UB, to write

$$\frac{|\eta_J(t)|}{\sqrt{m_2(t)}} \leq e^{-(\delta-\delta_0)(t-t_N)} \sum_{j=1}^{N} e^{-(\delta-\delta_0)(t_N - t_j)} [K + \varepsilon(t, t_0)] \; ; \quad t_N \leq t$$

where N is the total number of discontinuities in $[t_0, t]$, and ε is a term due to the initial conditions in x_F. Hence, under Assumption 3.6, we have that $\eta_J/\sqrt{m_2}$ is UB and integrating inside the intervals (t_j, t_{j+1}), $j = 1, 2, \ldots, N$ we get

$$\int_{t_I}^{t_I + T_I} \frac{\eta_J^2(t)}{m_2(t)} \, dt \leq C_0 + K_J'^2 \nu T_I$$

where K_J' is a constant depending on the size of the jumps in the plant parameters and their derivatives but independent of the initial conditions. It is now quite straightforward to verify the inequalities of the corollary, working as in the previous cases. Notice that although it is not difficult to keep the various Cauchy constants in the equations, resulting in sharper bounds, the inequalities tend to become rather messy. Moreover, the bounds related to the perturbation part of the plant and the parameter discontinuities are quite conservative, as they involve norms of —often sparse— perturbation matrices. And since the parameters μ', ν are typically expected to be very small, there is little intuition to be gained from such an approach. On the other hand, we must keep track of the adaptation parameters which eventually affects the upper bounds of μ, ν and μ' for the BIBO stability of the adaptive closed-loop system. And although the overall stability problem becomes very complicated at the final stage of the analysis, these bounds reveal some design guidelines which may be useful in various special cases. □□

Proof of Corollary 6.18:

Working as in Theorem 6.6 we first establish the existence and uniqueness of an absolutely continuous $\hat{\theta}$ satisfying the estimator ODE, at least in a subinterval of $[t_0, t_0 + T]$. Next, consider the positive definite function $V = \frac{1}{2\gamma} \hat{\phi}^\top \hat{\phi}$. Then, for $t \in (t_j, t_{j+1})$ and inside the subinterval where the solution exists

$$\dot{V} \leq -\sigma V - \frac{\epsilon_1^2 - \bar{\eta}^2}{2m_2} + \frac{1}{2\sigma\gamma} (\sigma K_{\theta *} + \mu)^2 \tag{6.24}$$

where $\bar{\eta} = \eta_s + \tilde{\eta} + \eta_J + \varepsilon$ Further, from the proof of Corollary 6.16 we have that for μ', ν sufficiently small so that $||x_F||^2/m_2$ is UB,

$$\frac{\bar{\eta}^2(t)}{m_2(t)} \leq O\left(\mu^2, \mu'^2\right) + K_J^2 e^{-2\delta_0(t - t_j)} + \varepsilon(t, t_0)$$

inside the interval (t_j, t_{j+1}). Integrating both sides of (6.24) inside (t_j, t_{j+1}) with the initial condition $V(t_j^+)$ we obtain

$$\begin{aligned} V(t) &\leq V(t_j^+)e^{-\sigma(t-t_j)} + O\left(\frac{K_J^2}{\sigma}\right) e^{-\beta(\sigma)(t-t_j)} + \ldots \\ &+ O\left(\frac{K_{\theta*}^2}{\gamma}, \frac{\mu^2}{\gamma\sigma^2}, \frac{\mu^2}{\sigma}, \frac{\mu'^2}{\sigma}\right) + \varepsilon(t, t_0) \end{aligned}$$

where $\beta < \min[\sigma, 2\delta_0]$ is a positive constant. Observing that $|V(t_j^+) - V(t_j^-)| \leq O\left(K_J^2/\gamma\right)$, it now follows that for any constant A there exists $\nu_1 > 0$ sufficiently small such that $\forall \nu \in [0, \nu_1]$

$$O\left(\frac{K_J^2}{\sigma}\right) e^{-\beta(\sigma)/\nu} \ , \ O\left(\frac{K_J^2}{\gamma}\right) e^{-\sigma/\nu} \leq A$$

Hence, for ν sufficiently small, $V(t_j)$ is a bounded sequence from which the first part of the corollary follows by extending the solutions to the whole interval $[t_0, t_0 + T]$.

Finally, properties (1) and (2) are obtained by following the same procedure as in the proof of Corollary 6.16. □□