EEE 586: TEST 1 NAME:______ SOLUTIONS

Open-Book only

Problem 1

1. Determine whether the function $f : \mathbf{R} \mapsto \mathbf{R},\ f(x) = x^2|x|$ is
   A. differentiable,
   B. locally Lipschitz continuous,
   C. globally Lipschitz continuous.

$\frac{df}{dx} = 2x|x| + x^2\text{sign}(x)$ which is continuous and bounded on compact sets. But it is unbounded on $\mathbf{R}$. Hence, A: Yes, B: Yes, C: No.

Problem 2

2. Determine whether the following functions are
   A. positive definite,
   B. positive semi-definite,
   C. radially unbounded

$$\begin{aligned} f_1(x_1, x_2) &= x_1^2 + x_1^2 x_2^4 \\ f_2(x_1, x_2) &= (x_1 + 2x_2)^2 + (x_1 - x_2)^2 \end{aligned}$$

f_1: Not positive definite since it is zero for $x_1 = 0, x_2 \neq 0$. It is positive semidefinite. It is not radially unbounded for the same reason as A ($x_2 \to \infty$ does not imply $f \to \infty$).

$f : 2$: Let $z = Ax$, where

$$A = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$$

Then $f_2(x(z)) = \|z\|^2$ which is obviously positive definite (and semidefinite) and radially unbounded in z. Since the map $z \mapsto x$ is invertible, it follows that f_2 has the same properties in x as well.

Problem 3

Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1^3 - 2x_2^3 \end{aligned}$$

1. Find and characterize all the equilibrium points.
2. Use a suitable Lyapunov function to show that the origin is a uniformly stable equilibrium.

1. At an equilibrium, $x_2 = 0$ and $x_1 = 0$. The Jacobian at the equilibrium is

$$J = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

which has two eigenvalues at 0 so the equilibrium is degenerate.

2. The degenerate equilibrium means that a quadratic Lyapunov function candidate based on the linearization will not be successful. This example has the general form of the nonlinear spring problem so we start with

$$V_1 = \frac{1}{2}x_2^2 \Rightarrow \dot{V}_1 = -2x_2^4 - x_2 x_1^3$$

Now add an x_1^n term to cancel the indefinite quantity:

$$V = \frac{1}{2}x_2^2 + \frac{1}{4}x_1^4 \Rightarrow \dot{V} = -2x_2^4 \leq 0$$

Hence, the zero equilibrium is uniformly stable.

Problem 4

Consider the system

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -2x_1 - (x_1^2 + x_2^2 - 1)x_2\end{aligned}$$

1. Find and characterize all the equilibrium points.
2. Show that the system has a periodic orbit.
3. Based on your analysis, provide an estimate (or a clear method to compute one) of the smallest set containing the periodic orbit.

1. At an equilibrium, $x_2 = 0$ and $x_1 = 0$. The Jacobian at the equilibrium is

$$J = \begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix}$$

which has eigenvalues at $\frac{1}{2}(1 \pm \sqrt{-7})$. So, the equilibrium is an unstable focus.

2. Use a similar technique as in Problem 3 (but looking at the properties away from the origin) and select

$$V = \frac{1}{2}x_2^2 + x_1^2 \Rightarrow \dot{V} = -(x_1^2 + x_2^2 - 1)x_2^2 \leq 0, \text{ for } x_1^2 + x_2^2 \geq 1$$

Hence, the level set $\{x|V(x) \leq c\}$ containing the unit circle $x_1^2 + x_2^2 = 1$ (so $\dot{V}(x) \leq 0$ outside that level set), is positively invariant. By the Poincare-Bendixson theorem, and in view of part 1, the system has a periodic orbit.

3. The ellipse $\{x|V(x) = c\}$ goes through the points $(x_1 = \sqrt{c}, x_2 = 0)$ and $(x_1 = 0, x_2 = \sqrt{2c})$. Hence, the smallest level set containing the unit circle is for $c = 1$. This is an "upper bound" estimate of the periodic orbit.

We can also get a "lower bound" for the periodic orbit by noting that $\dot{V}(x) \geq 0$ inside the unit circle $x_1^2 + x_2^2 = 1$. (This is because of the fortuitous choice of V.) This implies that the exterior of the level set $\{x|V(x) \geq c\}$ that is completely contained inside the unit circle, is also an invariant set. For this, $c = 1/2$. It now follows that the periodic orbit is in the set

$$\{x|V(x) \leq 1\} \cap \{x|V(x) \geq \frac{1}{2}\}$$

Problem 5

Consider the system

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - 2x_2\end{aligned}$$

1. Use Lyapunov methods to show that the origin is an asymptotically stable equilibrium.
2. Estimate the rate of exponential convergence of the states to zero.

Here, asymptotic stability is equivalent with the existence of a PD matrix P solving the Lyapunov equation $A^\top P + PA = -I$ where

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$$

Performing the calculations we have $-2p_{12} = -1, 2p_{12} - 4p_{22} = -1, -p_{22} + p_{11} - 2p_{12} = 0$, yielding

$$P = \begin{bmatrix} 1.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$$

Calculating the determinants of the principal minors (Hurwitz test) we have $1.5 > 0, det(P) = 0.75 - 0.25 > 0$ so P is PD and, hence, the zero equilibrium is asymptotically stable.

2. For an estimate of the convergence rate, we use the comparison theorem: If $\dot{V} < -aV$, then $V(t) \leq V(0)\exp(-at)$ and $|x(t)| \sim \exp(-[a/2]t)$. As derived in class, $a = \lambda_{min}(Q)/\lambda_{max}(P)$. Here, $a = 1/1.707$, so the exponential convergence rate of the states is at least $\exp(-t/3.4)$. This estimate is quite conservative since the computation of the eigenvalues of A shows that the states decay as $t\exp(-t)$.