CHAPTER 12 PROBLEMS

12.1 If $f(t) = e^{-at} \sin bt$, find **F**(s) using (a) the definition of the Laplace Transform and (b) the fact that $\mathcal{L}[e^{-at} f(t)] = \mathbf{F}(s + a)$.

12.2 Find f(t) if **F**(s) is given by the expression

$$\mathbf{F}(s) = \frac{24s}{(s+2)(s+4)(s+6)}$$

12.3 Find f(t) if **F**(s) is given by the expression

$$\mathbf{F}(s) = \frac{4(s+4)}{s(s^2+8s+20)}$$

12.4 Find f(t) if **F**(s) is given by the expression

$$\mathbf{F}(s) = \frac{12(s+2)}{(s^2+2s+1)(s+3)}$$

12.5 Given the function

$$\mathbf{F}(s) = \frac{24(s+10)}{s(s+2)(s+4)}$$

Find the initial and final values of the function by evaluating it in both the s-domain and time domain.

CHAPTER 12 SOLUTIONS

12.1 (a) By definition

$$\mathbf{F}(s) = \int_0^\infty f(t)\, e^{-st}\ dt$$

And since f(t) = e^{-at} sin bt

$$\mathbf{F}(s) = \int_0^\infty e^{-at} \sin bt\ e^{-st}\ dt$$

Using Euler's identity

$$\mathbf{F}(s) = \int_0^\infty e^{-(s+a)t} \left[\frac{e^{jbt} - e^{-jbt}}{2j}\right] dt$$
$$= \int_0^\infty \frac{e^{-(s+a-jb)t} - e^{-(s+a+jb)t}}{2j}\ dt$$

Evaluating the integral

$$= \frac{1}{2j}\left[\frac{1}{s+a-jb} - \frac{1}{s+a+jb}\right]$$
$$= \frac{b}{(s+a)^2 + b^2}$$

(b) In this case f(t) = sin bt. Then

$$\mathbf{F}(s) = \int_0^\infty e^{-st} \sin bt\ dt$$

Again, using the Euler identity

$$\mathbf{F}(s) = \int_0^\infty e^{-st} \left(\frac{e^{jbt} - e^{-jbt}}{2j}\right) dt$$
$$= \frac{1}{2j}\int_0^\infty \left(e^{-(s-jb)t} - e^{-(s+jb)t}\right) dt$$

Evaluating the integral

$$= \frac{1}{2j}\left[\frac{1}{s - jb} - \frac{1}{s + jb}\right]$$
$$= \frac{b}{s^2 + b^2}$$

Then using the fact that $\mathcal{L}[e^{-at} f(t)] = \mathbf{F}(s + a)$ where in this case f(t) = sin bt and

$$\mathbf{F}(s) = \frac{b}{s^2 + b^2}$$

we find that

$$\mathbf{F}(s + a) = \mathcal{L}\left[e^{-at}\ f(t)\right]$$
$$= \mathcal{L}\left[e^{-at}\ \sin bt\right]$$
$$= \frac{b}{(s + a)^2 + b^2}$$

12.2 The expression

$$\mathbf{F}(s) = \frac{24s}{(s + 2)(s + 4)(s + 6)}$$

can be written in a partial fraction expansion of the form

$$\frac{24s}{(s + 2)(s + 4)(s + 6)} = \frac{k_1}{s + 2} + \frac{k_2}{s + 4} + \frac{k_3}{s + 6}$$

Multiplying the entire equation by the term s + 2 yields

$$\frac{24s}{(s + 4)(s + 6)} = k_1 + \frac{k_2(s + 2)}{s + 4} + \frac{k_3(s + 2)}{s + 6}$$

If we now evaluate each term at s = -2, we find that the last two terms on the right side of the equation vanish and we have

$$\left.\frac{24s}{(s + 4)(s + 6)}\right|_{s = -2} = k_1$$
$$-6 = k_1$$

Repeating this procedure for the two remaining terms in the denominator, i.e., (s + 4) and (s + 6) yields

$$\left.\frac{24s}{(s+2)(s+6)}\right|_{s=-4} = k_2$$

$$24 = k_2$$

And

$$\left.\frac{24s}{(s+2)(s+4)}\right|_{s=-6} = k_3$$

$$-18 = k_3$$

Now the function **F**(s) can be written in the form

$$\mathbf{F}(s) = \frac{-6}{s+2} + \frac{24}{s+4} - \frac{18}{s+6}$$

The reader can check the validity of this expansion by recombining the terms to produce the original expression.

Once F(s) is in this latter form, we can use the transform pair

$$\mathcal{L}\left[e^{-at}\right] = \frac{1}{s+a}$$

And hence

$$f(t) = \left[-6e^{-2t} + 24e^{-4t} - 18e^{-6t}\right]u(t)$$

12.3 We begin by writing the function in a partial fraction expansion. Therefore, we need to know the roots of the quadratic term. We can either employ the quadratic formula or recognize that

$$\begin{aligned} s^2 + 8s + 20 &= s^2 + 8s + 16 + 4 \\ &= (s+4)^2 + 4 \\ &= (s + 4 - j2)\,(s + 4 + j2) \end{aligned}$$

Hence, the function F(s) can be written as

$$\mathbf{F}(s) = \frac{4(s+4)}{s(s+4-j2)(s+4+j2)} = \frac{k_0}{s} + \frac{k_1}{s+4-j2} + \frac{k_1^*}{s+4+j2}$$

Multiplying the entire equation by s and evaluating it at s = 0 yields

$$\left.\frac{4(s+4)}{s^2+8s+20}\right|_{s=0} = k_0$$

$$\frac{4}{5} = k_0$$

Using the same procedure for k_1, we obtain

$$\left.\frac{4(s+4)}{s(s+4+j2)}\right|_{s=-4+j2} = k_1$$

$$\frac{1}{-2+j} = k_1$$

$$\frac{-1}{2-j} = k_1$$

$$\frac{-(2+j)}{5} = k_1$$

$$\frac{1}{\sqrt{5}}\angle 206.56° = k_1$$

Then, we know that

$$\frac{1}{\sqrt{5}}\angle -206.56° = k_1^*$$

Now using the fact that

$$\mathsf{L}^{-1}\left[\frac{|k_1|\angle\theta}{s+a-jb}+\frac{|k_1|\angle-\theta}{s+a+jb}\right] = 2|k_1|\, e^{-at}\cos(bt+\theta)$$

The function f(t) is

$$f(t) = \left[\frac{4}{5}+\frac{2}{\sqrt{5}}e^{-4t}\cos(2t+206.56°)\right]u(t)$$

12.4 In order to perform a partial fraction expansion on the function **F**(s), we need to factor the quadratic term. We can use the quadratic formula or simply note that (s + 1) (s + 1) = s^2 + 2s + 1. Therefore, **F**(s) can be expressed as

$$\mathbf{F}(s) = \frac{12(s+2)}{(s+1)^2\,(s+3)}$$

or in the form

$$\mathbf{F}(s) = \frac{12(s+2)}{(s+1)^2\,(s+3)} = \frac{k_{11}}{s+1} + \frac{k_{12}}{(s+1)^2} + \frac{k_2}{s+3}$$

If we now multiply the entire equation by $(s + 1)^2$, we obtain

$$\frac{12(s+2)}{s+3} = k_{11}(s+1) + k_{12} + \frac{k_2\,(s+1)^2}{s+3}$$

Now evaluating this equation at s = -1 yields

$$\left.\frac{12(s+2)}{s+3}\right|_{s=-1} = k_{12}$$

$$6 = k_{12}$$

In order to evaluate k_{11} we differentiate each term of the equation with respect to s and evaluate all terms at s = -1. Note that the derivative of k_{12} with respect to s is zero, the derivative of the last term in the equation with respect to s will still have an (s + 1) term in the numerator that will vanish when evaluated at s = -1, and the derivative of the first term on the right side of the equation with respect to s simply yields k_{11}. Therefore,

$$\frac{d}{ds}\left[\frac{12(s+2)}{s+3}\right]_{s=-1} = k_{11}$$

$$\left.\frac{(s+3)(12) - 12(s+2)(1)}{(s+3)^2}\right|_{s=-1} = k_{11}$$

$$3 = k_{11}$$

Finally,

$$\left.\frac{12(s+2)}{(s+1)^2}\right|_{s=-3} = k_2$$

$$-3 = k_2$$

And therefore, **F**(s) can be expressed in the form

$$\mathbf{F}(s) = \frac{3}{s+1} + \frac{6}{(s+1)^2} - \frac{3}{s+2}$$

Using the transform pairs, we find that

$$f(t) = \left[3e^{-t} + 6te^{-t} - 3e^{-2t}\right]u(t)$$

12.5 First, let us use the Theorems to evaluate the function in the s-domain.

The initial value can be derived from the Theorem

$$\lim_{t \to 0} f(t) = \lim_{s \to \infty} s\mathbf{F}(s)$$

Therefore,

$$\begin{aligned}\lim s\mathbf{F}(s) &= \lim_{s \to \infty} \left[\frac{24(s+10)}{(s+2)(s+4)}\right] \\ &= \lim_{s \to \infty} \left[\frac{24s + 240}{s^2 + 6s + 8}\right] \\ &= \lim_{s \to \infty} \left[\frac{\frac{24}{s} + \frac{240}{s^2}}{1 + \frac{6}{s} + \frac{8}{s^2}}\right] \\ &= 0\end{aligned}$$

The final value is derived from the expression

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} s\mathbf{F}(s)$$

Hence,

$$\begin{aligned}\lim_{s \to 0} s\mathbf{F}(s) &= \lim_{s \to 0} \left[\frac{24(s+10)}{(s+2)(s+4)}\right] \\ &= \frac{240}{8} \\ &= 30\end{aligned}$$

The time function can be derived from a partial fraction expansion as

$$\mathbf{F}(s) = \frac{24(s+10)}{s(s+2)(s+4)} = \frac{k_0}{s} + \frac{k_1}{s+2} + \frac{k_2}{s+4}$$

where

$$\left.\frac{24(s+10)}{(s+2)(s+4)}\right|_{s=0} = k_0 = 30$$

$$\left.\frac{24(s+10)}{s(s+4)}\right|_{s=-2} = k_1 = -48$$

$$\left.\frac{24(s+10)}{s(s+2)}\right|_{s=-4} = k_2 = 18$$

Hence,

$$\mathbf{F}(s) = \frac{30}{s} - \frac{48}{s+2} + \frac{18}{s+4}$$

and then

$$f(t) = \left[30 - 48e^{-2t} + 18e^{-4t}\right]u(t)$$

Given this expression, we find that

$$\lim_{t \to 0} f(t) = [30 - 48 + 18] = 0$$

and

$$\lim_{t \to \infty} f(t) = [30 - 0 + 0] = 30$$