CHAPTER 13 PROBLEMS

13.1 Find $v_0(t)$, $t > 0$ in the circuit in Fig. 13.1 using (a) nodal analysis, (b) source transformation and (c) Norton's Theorem.

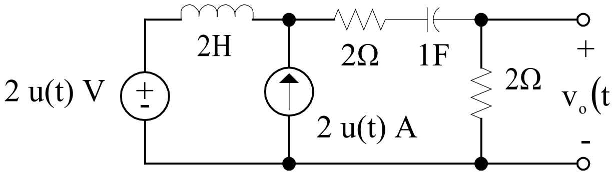

Fig. 13.1

13.2 Find $i_0(t)$, $t > 0$ in the circuit in Fig. 13.2 using (a) loop equations and (b) Thevenin's Theorem.

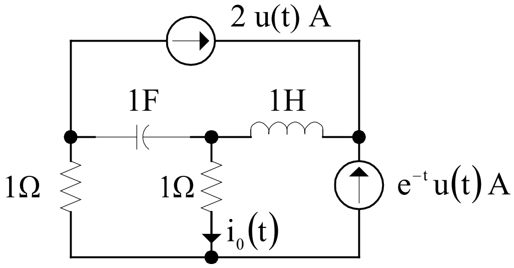

Fig. 13.2

13.3 Find $i_0(t)$, $t > 0$ in the circuit in Fig. 13.3.

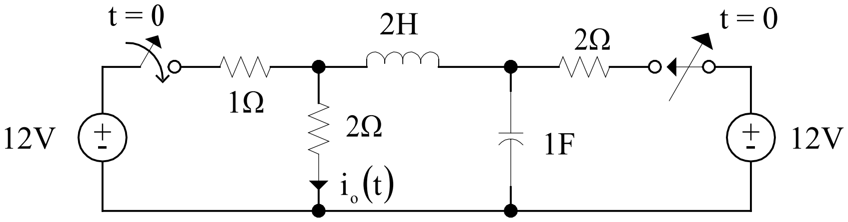

Fig. 13.3

13.4 Given the network in Fig. 13.4, determine (a) the voltage transfer function $\mathbf{G}(s) = \frac{\mathbf{V}_0(s)}{\mathbf{V}_s(s)}$, (b) the undamped natural frequency, (c) the damping ratio and (d) the general form of the response of the network to a unit step function.

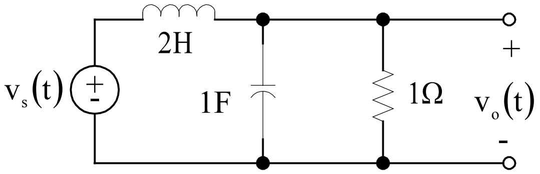

Fig. 13.4

13.5 Find the steady-state response $v_0(t)$ for the network in Fig. 13.5.

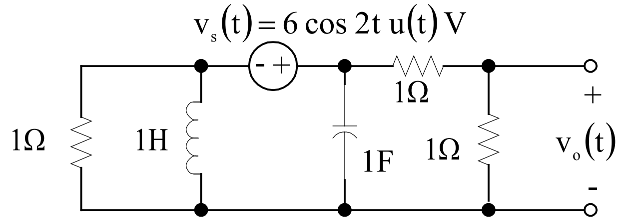

Fig. 13.5

CHAPTER 13 SOLUTIONS

13.1 (a) Consider the transformed network in Fig. S13.1(a).

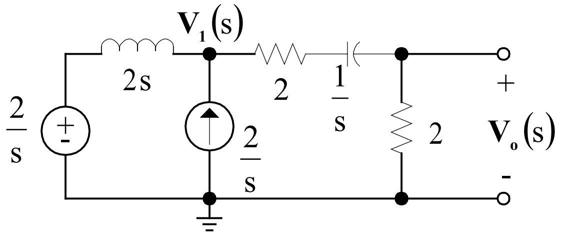

Fig. S13.1(a)

A brute force approach to this problem would be to write two nodal equations for the nodes labeled $\mathbf{V_1}(s)$ and $\mathbf{V_0}(s)$. Using KCL and summing the currents leaving each node yields the two linearly independent equations

$$\frac{\mathbf{V_1}(s) - \frac{2}{s}}{2s} - \frac{2}{s} + \frac{\mathbf{V_1}(s) - \mathbf{V_0}(s)}{2 + \frac{1}{s}} = 0$$

and

$$\frac{\mathbf{V_0}(s) - \mathbf{V_1}(s)}{2 + \frac{1}{s}} + \frac{\mathbf{V_0}(s)}{2} = 0$$

Solving these equations for $\mathbf{V_0}(s)$ and then performing the inverse Laplace transform would yield $v_0(t)$.

Another approach that might be simpler would be to write a node equation for $\mathbf{V_1}(s)$, ignoring $\mathbf{V_0}(s)$, and then use voltage division to derive $\mathbf{V_0}(s)$ once $\mathbf{V_1}(s)$ is known. Applying KCL at $\mathbf{V_1}(s)$ yields

$$\frac{\mathbf{V_1}(s) - \frac{2}{s}}{2s} - \frac{2}{s} + \frac{\mathbf{V_1}(s)}{4 + \frac{1}{s}} = 0$$

Rearranging terms we obtain

$$\mathbf{V_1}(s)\left[\frac{1}{2s} + \frac{s}{4s+1}\right] = \frac{1}{s^2} + \frac{2}{s}$$

or

$$\mathbf{V}_1(s)\left[\frac{2s^2+4s+1}{2s(4s+1)}\right]=\frac{2s+1}{s^2}$$

Solving for $\mathbf{V}_1(s)$ yields

$$\mathbf{V}_1(s)=\frac{2(2s+1)(4s+1)}{s(2s^2+4s+1)}$$

Now applying voltage division

$$\mathbf{V}_0(s)=\mathbf{V}_1(s)\left(\frac{2}{4+\frac{1}{s}}\right)$$
$$=\frac{4(2s+1)}{2s^2+4s+1}$$

This function can be written in partial fraction expansion form as

$$\frac{4s+2}{s^2+2s+\frac{1}{2}}=\frac{A}{s+0.29}+\frac{B}{s+1.71}$$

where

$$A=\left.\frac{4s+2}{s+1.71}\right|_{s\ =\ -0.29}=0.59$$

and

$$B=\left.\frac{4s+2}{s+0.29}\right|_{s\ =\ -1.71}=3.41$$

Therefore,

$$v_0(t)=\left[0.59e^{-0.29t}+3.41e^{-1.71t}\right]u(t)\text{ V}$$

(b) Using source transformation we can convert the voltage source in series with the inductor to a current source in parallel with the inductor yielding the network in Fig. S13.1(b).

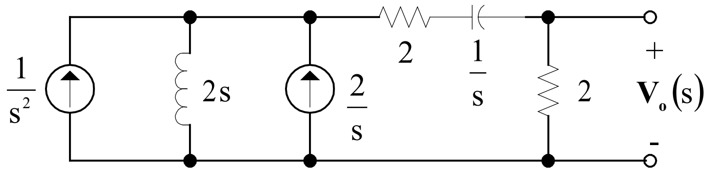

Fig. S13.1(b)

Adding the current sources that are in parallel produces an equivalent source of

$$\mathbf{I}_{EQ}(s) = \frac{1}{s^2} + \frac{2}{s} = \frac{2s+1}{s^2}$$

The network is then reduced to that shown in Fig. S13.1(c).

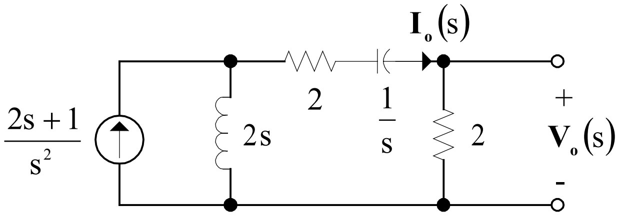

Fig. S13.1(c)

We could, at this point, transform the current source and inductor back to a voltage source in series with the inductor. However, we can simply apply current division at this point with Ohm's Law and derive the answer immediately.

$$\mathbf{I}_0(s) = \frac{2s+1}{s^2}\left(\frac{2s}{2s+2+\frac{1}{s}+2}\right)$$

$$= \frac{4s+2}{2s^2+4s+1}$$

And

$$\mathbf{V}_0(s) = 2\mathbf{I}_0(s) = \frac{4s+2}{s^2+2s+\frac{1}{2}}$$

which is identical to the expression obtained earlier.

(c) To apply Norton's Theorem we will break the network to the right of the current source and form a Norton equivalent circuit for the elements to the left of the break as shown in Fig. S13.1(d).

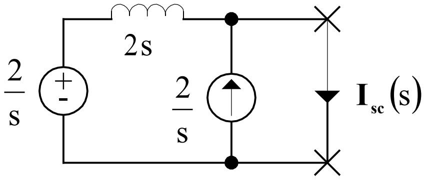

Fig. S13.1(d)

The short-circuit current is

$$\mathbf{I}_{sc}(s) = \frac{\frac{2}{s}}{2s} + \frac{2}{s}$$

$$= \frac{2s+1}{s^2}$$

And the Thevenin equivalent impedance is derived from the network in Fig. S13.1(e) as

$$\mathbf{Z}_{TH}(s) = 2s$$

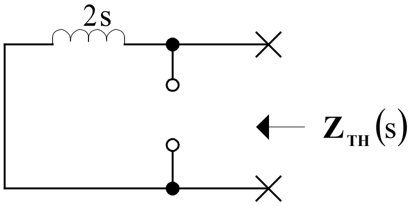

Fig. S13.1(e)

Therefore, attaching the Norton equivalent circuit to the remainder of the network yields the circuit in Fig. S13.1(f) which is the same as that in Fig. S13.1(c).

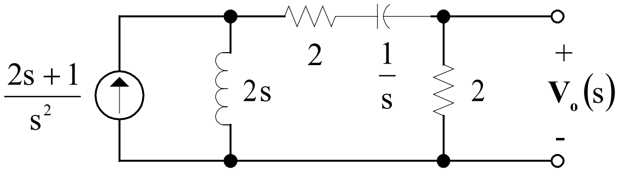

Fig. S13.1(f)

13.2 (a) the transformed network is shown in Fig. S13.2(a).

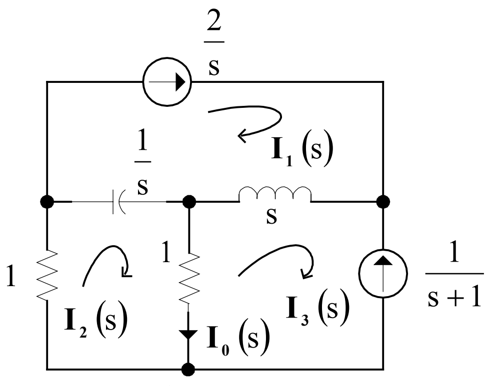

Fig. S13.2(a)

Since there are three "window panes" we will need three linearly independent simultaneous equations to calculate the loop currents. Two of the currents go directly through the current sources and therefore two of the three equations are

$$\mathbf{I}_1(s) = \frac{2}{s}$$

$$\mathbf{I}_3(s) = \frac{-1}{s+1}$$

The remaining equation is obtained by using KVL around the loop defined by the current $\mathbf{I}_2$(s). That equations is

$$1\mathbf{I}_2(s) + \frac{1}{s}\left[\mathbf{I}_2(s) - \mathbf{I}_1(s)\right] + 1\left[\mathbf{I}_2(s) - \mathbf{I}_3(s)\right] = 0$$

Substituting the first two equations into the last equation yields

$$\mathbf{I}_2(s)\left[1 + \frac{1}{s} + 1\right] = \frac{2}{s^2} - \frac{1}{s+1}$$

or

$$\mathbf{I}_2(s) = \frac{-s^2 + 2s + 2}{s(s+1)(2s+1)}$$

Then

$$
\begin{aligned}
\mathbf{I}_0(s) &= \mathbf{I}_2(s) - \mathbf{I}_3(s) \\
&= \frac{-s^2 + 2s + 2}{s(s+1)(2s+1)} + \frac{1}{s+1} \\
&= \frac{s^2 + 3s + 2}{s(s+1)(2s+1)} \\
&= \frac{s+2}{s(2s+1)} \\
&= \frac{\frac{1}{2}(s+2)}{s\left(s+\frac{1}{2}\right)}
\end{aligned}
$$

Expressing this function in partial fraction expansion form we obtain

$$\mathbf{I}_0(s) = \frac{\frac{1}{2}(s+2)}{s\left(s+\frac{1}{2}\right)} = \frac{A}{s} + \frac{B}{s+\frac{1}{2}}$$

where

$$A = \left.\frac{\frac{1}{2}(s+2)}{s+\frac{1}{2}}\right|_{s=0} = 2$$

$$B = \left.\frac{\frac{1}{2}(s+2)}{s}\right|_{s=-\frac{1}{2}} = -\frac{3}{2}$$

Therefore,

$$i_0(t) = \left[2 - \frac{3}{2}e^{-\frac{t}{2}}\right]u(t)\ \text{A}$$

(b) In order to apply Thevenin's Theorem, we first break the circuit between the points where the current $\mathbf{I}_0(s)$ is located as shown in Fig. S13.2(b).

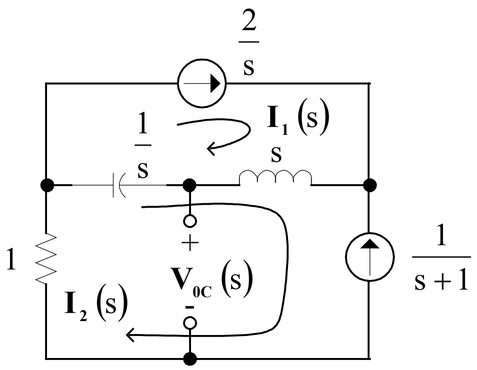

Fig. S13.2(b)

Applying KVL to the closed path in the lower left-hand corner of the network yields

$$1\mathbf{I}_2(s)+\frac{1}{s}[\mathbf{I}_2(s)-\mathbf{I}_1(s)]+\mathbf{V}_{0C}(s)=0$$

where

$$\mathbf{I}_1(s)=\frac{2}{s}$$

$$\mathbf{I}_2(s)=\frac{-1}{s+1}$$

Combining these equations we obtain

$$\mathbf{V}_{0C}(s)=\frac{1}{s+1}+\frac{1}{s(s+1)}+\frac{2}{s^2}$$

$$=\frac{s+2}{s^2}$$

The Thevenin equivalent impedance obtained by looking into the open circuit terminals with all sources made zero (current sources open-circuited) is derived from the network in Fig. S13.2(c).

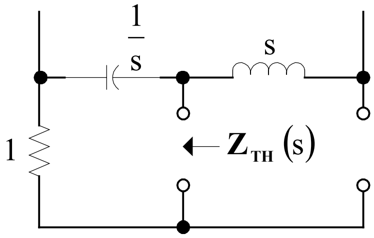

Fig. S13.2(c)

Clearly,

$$\mathbf{Z}_{\mathbf{TH}}(s)=\frac{1}{s}+1=\frac{s+1}{s}$$

If the resistor containing the $\mathbf{I}_0(s)$ is now attached to the Thevenin equivalent circuit we obtain the network in Fig. S13.2(d).

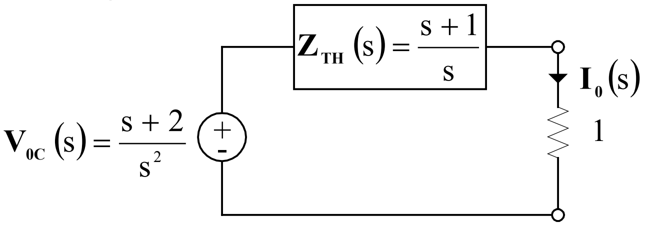

Fig. S13.2(d)

Then

$$\mathbf{I}_0(s)=\frac{\dfrac{s+2}{s^2}}{\dfrac{s+1}{s}+1}$$

$$=\frac{s+2}{s(2s+1)}$$

which is identical to the result obtained earlier.

13.3 To begin, we first determine the initial conditions in the network prior to switch action. In the steady-state period prior to switch action, the capacitor looks like an open-circuit and the inductor acts like a short-circuit. Therefore, in this time interval the circuit appears as that shown in Fig. S13.3(a).

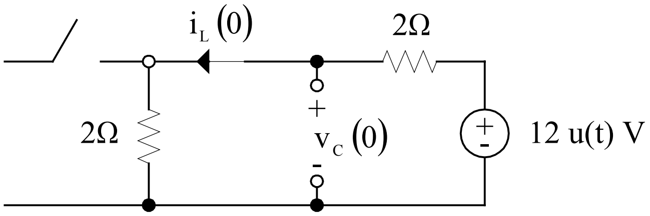

Fig. S13.3(a)

This network indicates that in the steady-state condition for t < 0

$$i_L(0)=\frac{12}{2+2}=3\text{A}$$

and

$$v_C(0) = 12\left(\frac{2}{2+2}\right) = 6\text{V}$$

These conditions cannot change instantaneously and hence the network for t > 0 is shown in Fig. S13.3(b).

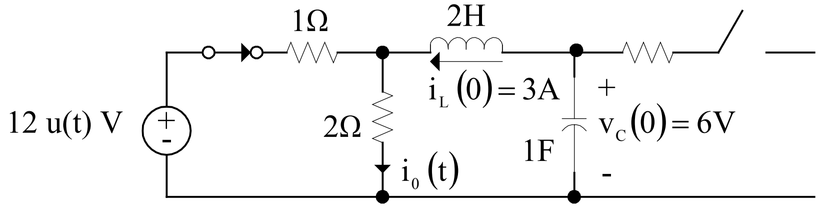

Fig. S13.3(b)

The corresponding transformed network is shown in Fig. S13.3(c).

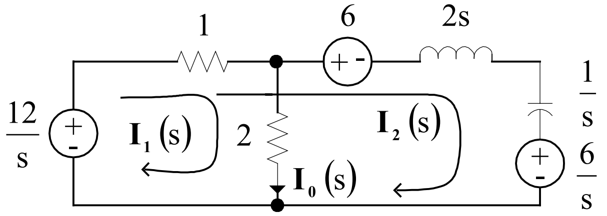

Fig. S13.3(c)

Since the current $\mathbf{I}_0(s)$ is located in the center leg of the circuit, we will employ loop equations and specify them such that one of the loops is the same as $\mathbf{I}_0(s)$. The two equations for the loop currents specified in the network are

$$\frac{-12}{s} + 1(\mathbf{I}_1(s) + \mathbf{I}_2(s)) + 2\mathbf{I}_1(s) = 0$$

$$\frac{-12}{s} + 1(\mathbf{I}_1(s) + \mathbf{I}_2(s)) + 6 + 2s\mathbf{I}_2(s) + \frac{1}{s}\mathbf{I}_2(s) + \frac{6}{s} = 0$$

Solving the second equation for $\mathbf{I}_2(s)$ yields

$$\mathbf{I}_2(s) = \frac{11 - 6s - s\mathbf{I}_1(s)}{2s^2 + s + 1}$$

Substituting this value into the first equation we obtain

$$\mathbf{I}_1(s) = \mathbf{I}_0(s) = \frac{\frac{1}{6}(30s^2 + s + 12)}{s\left(s^2 + \frac{1}{3}s + \frac{1}{2}\right)}$$

The roots of the quadratic term in the denominator, obtained using the quadratic formula, are

$$s_1, s_2 = -\frac{1}{6} \pm j\frac{\sqrt{17}}{6}$$

The expression for the desired current can now be written in partial fraction expansion form as

$$\frac{\frac{1}{6}\left(30s^2 + s + 12\right)}{s\left(s + \frac{1}{6} \pm j\frac{\sqrt{17}}{6}\right)} = \frac{A}{s} + \frac{B}{s - \frac{1}{6} + j\frac{\sqrt{17}}{6}} + \frac{B^*}{s + \frac{1}{6} + j\frac{\sqrt{17}}{6}}$$

where

$$\left.\frac{\frac{1}{6}\left(30s^2 + s + 12\right)}{s^2 + \frac{1}{3}s + \frac{1}{2}}\right|_{s = 0} = A$$

$$4 = A$$

and

$$\left.\frac{\frac{1}{6}\left(30s^2 + s + 12\right)}{s\left(s + \frac{1}{6} + j\frac{\sqrt{17}}{6}\right)}\right|_{s = -\frac{1}{6} + j\frac{\sqrt{17}}{6}} = B$$

The evaluation of this last term involves a lot of tedious, but straight forward, complex algebra. The result is

$$1.09\ \angle 62.74° = B$$

Therefore, knowing the values for A and B we can write the final expression for the current in the time domain as

$$i_0(t) = \left[4 + 2(1.09)\,e^{-\frac{t}{6}}\cos\left(\frac{\sqrt{17}}{6}t + 62.74°\right)\right]u(t)\ A$$

13.4 (a) The transformed network is shown in Fig. S13.4.

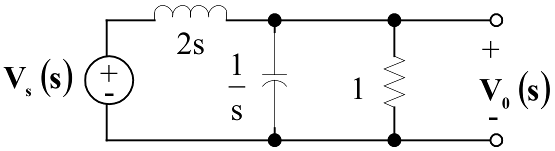

Fig. S13.4

Using voltage division, the voltage transfer function can be expressed as

$$\mathbf{G}(s) = \frac{\mathbf{V}_0(s)}{\mathbf{V}_s(s)} = \frac{\dfrac{2\left(\frac{1}{s}\right)}{2+\frac{1}{s}}}{2s + \dfrac{2\left(\frac{1}{s}\right)}{2+\frac{1}{s}}}$$

$$= \frac{2}{4s^2 + 2s + 2}$$

$$= \frac{\frac{1}{2}}{s^2 + \frac{1}{2}s + \frac{1}{2}}$$

(b) The denominator, or characteristic equation, is of the form

$$s^2 + 2\zeta\omega_0 s + \omega_0^2$$

Therefore the undamped natural frequency is

$$\omega_0^2 = \frac{1}{2}$$

and

$$\omega_0 = \frac{1}{\sqrt{2}} = 0.707 \text{ r/s}$$

(c) The damping ratio is derived from the expression

$$2\zeta\omega_0 = \frac{1}{2}$$

and using the value for ω_0 we obtain

$$\zeta = 0.354$$

(d) If the input to the network is a unit step function then

$$\mathbf{V}_0(s) = \frac{\frac{1}{2}}{s\left(s^2 + \frac{1}{2}s + \frac{1}{2}\right)}$$

By employing the quadratic formula, we can write this expression in the form

$$\mathbf{V}_0(s) = \frac{\frac{1}{2}}{s\left(s + \frac{1}{4} \pm j\frac{\sqrt{7}}{4}\right)}$$

and therefore the general form of the response is

$$v_0(t) = \left[A + B\,e^{-\frac{1}{4}t}\cos\left(\frac{\sqrt{7}}{4}t + \theta\right)\right]u(t)\ \mathrm{v}$$

13.5 The transformed circuit is shown in Fig. S13.5.

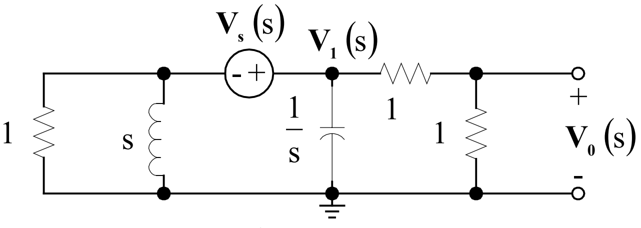

Fig. S13.5

Although the network contains three non-reference nodes, we will try to simplify the analysis by first using a supernode to find **V₁**(s) and then employing voltage division to determine $\mathbf{V}_0(s)$.

KCL for the supernode containing the voltage source is

$$\frac{\mathbf{V}_1(s) - \mathbf{V}_s(s)}{1} + \frac{\mathbf{V}_1(s) - \mathbf{V}_s(s)}{s} + \frac{\mathbf{V}_1(s)}{\frac{1}{s}} + \frac{\mathbf{V}_1(s)}{2} = 0$$

Solving this equation for $\mathbf{V}_1(s)$ yields

$$\mathbf{V}_1(s) = \left(\frac{s+1}{s^2 + \frac{3}{2}s + 1}\right)\mathbf{V}_s(s)$$

And then using voltage division

$$\mathbf{V}_0(s) = \mathbf{V}_1(s)\left(\frac{1}{1+1}\right)$$

so that

$$\mathbf{V}_0(s) = \left[\frac{\frac{1}{2}(s+1)}{s^2 + \frac{3}{2}s + 1}\right]\mathbf{V}_s(s)$$

Therefore,

$$\mathbf{H}(s) = \frac{\frac{1}{2}(s+1)}{s^2 + \frac{3}{2}s + 1}$$

Since $v_s(t) = 6 \cos 2t\, u(t)$ V, then $V_M = 6$ and $\omega = 2$. Hence,

$$\begin{aligned}\mathbf{H}(j2) &= \frac{\frac{1}{2}(j2+1)}{(j2)^2 + \frac{3}{2}(j2) + 1}\\ &= \frac{-\frac{1}{2}(2.236\angle 63.43°)}{4.24\angle -45°}\\ &= 0.264\angle -71.57°\end{aligned}$$

and

$$\begin{aligned}|\mathbf{H}(j2)| &= 0.264\\ \phi(j2) &= -71.57°\end{aligned}$$

Therefore,

$$\begin{aligned} v_{0ss}(t) &= V_M \left| \mathbf{H}(j2) \right| \cos\left(2t + \phi(j2)\right) \\ &= 1.58 \cos\left(2t - 71.57^\circ\right) \text{V} \end{aligned}$$