CHAPTER 14 PROBLEMS

14.1 Find the exponential Fourier series for the waveform in Fig. 14.1.

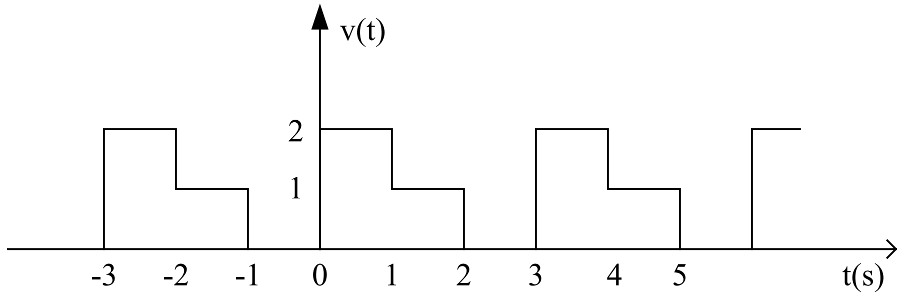

Fig. 14.1

14.2 Determine the trigonometric Fourier series for the function shown in Fig. 14.2.

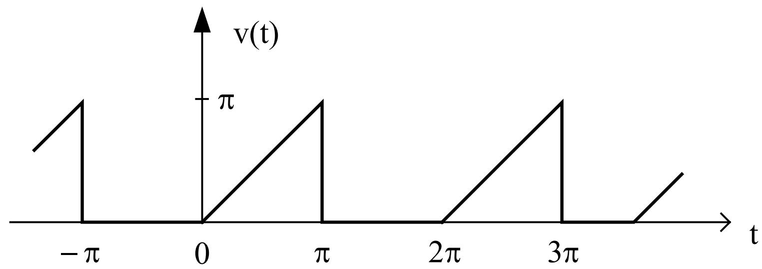

Fig. 14.2

14.3 Find the trigonometric Fourier series for the waveform shown in Fig. 14.3.

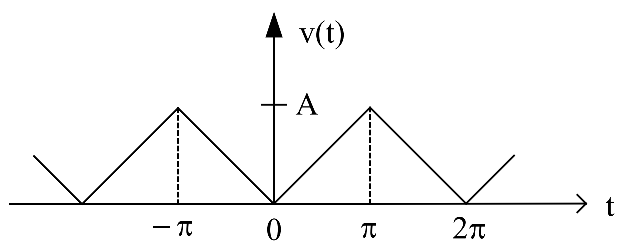

Fig. 14.3

14.4 Find the steady-state voltage $v_0(t)$ in the circuit in Fig. 14.4 if the input voltage is the waveform shown in Fig. 14.3 with A = 1V.

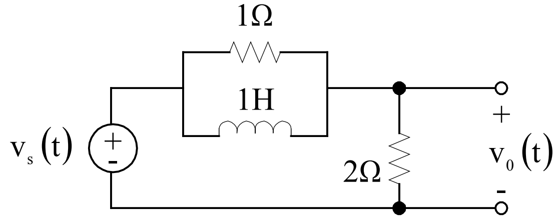

Fig. 14.4

14.5 Given the network in Fig. 14.4 with the input source $v_s(t) = 10\,e^{-2t}\,u(t)\,V$, use the transform technique to find $v_0(t)$.

CHAPTER 14 SOLUTIONS

14.1 An examination of the waveform indicates that the period T = 3 and $\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{3}$.

The Fourier coefficients are determined from the expression

$$c_n = \frac{1}{T}\int_0^T v(t)\, e^{-jn\omega_0 t}\, dt$$

or in this case

$$\begin{aligned}
c_n &= \frac{1}{3}\left[\int_0^1 2\ e^{-jn\omega_0 t}\ dt + \int_1^2 1\, e^{-jn\omega_0 t}\ dt\right] \\
&= \frac{-1}{3jn\omega_0}\left[2\, e^{-jn\omega_0 t}\Big|_0^1 + e^{-jn\omega_0 t}\Big|_1^2\right] \\
&= \frac{-1}{3jn\omega_0}\left[2\left(e^{-jn\omega_0} - 1\right) + e^{-j2n\omega_0} - e^{-jn\omega_0}\right] \\
&= \frac{-1}{3jn\omega_0}\left[e^{-jn\omega_0} + e^{-j2n\omega_0} - 2\right] \\
&= \frac{-1}{j2\pi n}\left[e^{\frac{-j2\pi n}{3}} + e^{\frac{-j4\pi n}{3}} - 2\right] \\
&= \frac{1}{j2\pi n}\left(2 - \left(e^{\frac{-j2\pi n}{3}} + e^{\frac{-j4\pi n}{3}}\right)\right) \\
&= \frac{1}{j2\pi n}\left[2 - 2\, e^{-j\pi n}\left(\frac{e^{\frac{jn\pi}{3}} + e^{\frac{-jn\pi}{3}}}{2}\right)\right] \\
&= \frac{1}{jn\pi}\left(1 - e^{-jn\pi}\cos\left(\frac{n\pi}{3}\right)\right)
\end{aligned}$$

In addition

$$\begin{aligned}
c_0 &= \frac{1}{T}\int_0^3 v(t)\, dt \\
&= \frac{1}{3}\left[\int_0^1 2dt + \int_1^2 1dt\right] \\
&= 1
\end{aligned}$$

Therefore,

$$v(t) = 1 + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{1}{jn\pi}\left(1 - e^{-jn\pi} \cos\left(\frac{n\pi}{3}\right)\right) e^{jn\omega_0 t}$$

14.2 Since the waveform does not exhibit any symmetry, we will have to determine the coefficients a_0, a_n and b_n. The a_0 coefficient is

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} v(t)\, dt$$

where, of course, v(t) = t in the interval $0 \le t \le \pi$ and zero elsewhere and $\omega_0 = \frac{2\pi}{T} = 1$.

$$a_0 = \frac{1}{2\pi} \int_0^{\pi} t\, dt$$
$$= \frac{1}{2\pi}\left(\left.\frac{t^2}{2}\right|_0^{\pi} = \frac{\pi}{4}\right)$$

Recall that a_0 is simply the average value of the waveform and therefore can be calculated by dividing the area under the curve ($\text{Area} = \frac{1}{2}\,bh = \frac{1}{2}(\pi)(\pi) = \frac{\pi^2}{2}$) by the interval (2π) which yields $\frac{\pi}{4}$.

The a_n coefficient is

$$a_n = \frac{2}{2\pi} \int_0^{\pi} t \cos nt\, dt$$

Using a table of integrals, we find that

$$a_n = \frac{1}{\pi}\left[\frac{1}{n^2} \cos nt + \frac{1}{n} t \sin nt\right]_0^{\pi}$$

The second term is zero at t = π and 0 and the first term can be written as

$$a_n = \frac{1}{\pi n^2}\left[(-1)^n - 1\right]$$

since the cosine term will be +1 or –1 depending upon the value of n. Thus

$$a_n = \frac{(-1)^n - 1}{\pi n^2}$$

In addition,

$$b_n = \frac{2}{2\pi}\int_0^{\pi} t \sin t \, dt$$

Once again, using a set of integral tables we find that

$$b_n = \frac{1}{\pi}\left[\frac{1}{n^2}\sin nt - \frac{1}{n} t \cos nt\right]_0^{\pi}$$

The first term will be zero at each limit, but the second term is nonzero at the upper limit and thus

$$b_n = \frac{-\pi}{n\pi}(-1)^n$$
$$= \frac{-(-1)^n}{n}$$

Therefore, the Fourier series expansion is

$$v(t) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{\pi n^2}\right]\cos nt - \frac{(-1)^n}{n}\sin nt$$

14.3 To begin our analysis we first note that the waveform is an even function and therefore $b_n = 0$ for all n. Thus, we need to find only the a_0 and a_n coefficients.

For this waveform, we note that T = 2π and $\omega_0 = \frac{2\pi}{T} = 1$. a_0 is now

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} v(t)\, dt$$

However, recall that a_0 is simply the average value and we can easily compute this number without resorting to solving the above integral. This average value can be obtained by dividing the area by the base, i.e.

$$\text{Area} = 2\left(\frac{1}{2}\, bh\right) = 2\left(\frac{1}{2}(\pi A)\right) = \pi A$$

The base is 2π and therefore

$$a_0 = \frac{\pi A}{2\pi} = \frac{A}{2}$$

Because the function is even,

$$a_n = \frac{4}{2\pi}\int_0^{\pi} \frac{A}{\pi} t \cos nt \, dt$$

where the equation of the straight line function in the interval $0 \le t \le \pi$ is $\frac{A}{\pi}t$. So,

$$a_n = \frac{2A}{\pi^2}\int_0^{\pi} t \cos nt \, dt$$

Using a table of integrals, we find that

$$\begin{aligned}
a_n &= \frac{2A}{\pi^2}\left[\frac{1}{n^2}\cos nt + \frac{1}{n} t \sin nt\right]_0^{\pi} \\
&= \frac{2A}{\pi^2}\left[\frac{1}{n^2}(\cos n\pi - 1)\right] \\
&= \frac{2A}{(\pi n)^2}(\cos n\pi - 1) \\
&= \frac{-4A}{(\pi n)^2} \qquad \text{for n odd} \\
&= 0 \qquad\qquad \text{for n even}
\end{aligned}$$

Therefore,

$$v(t) = \frac{A}{2} + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{-4A}{(\pi n)^2} \cos nt$$

14.4 The input voltage for the circuit in Fig. 14.4 is given by the expression

$$v_s(t) = \frac{1}{2} + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{-4}{(\pi n)^2} \cos nt$$

where $\omega_0 = 1$. The output voltage for the network can be derived using voltage division as

$$\mathbf{V_0}(j\omega) = \frac{2}{2 + \frac{(1)(j\omega)}{1 + j\omega}} \mathbf{V_s}(j\omega)$$

$$= \left[\frac{2(1 + j\omega)}{2 + 3j\omega}\right] \mathbf{V_s}(j\omega)$$

and since $\omega_0 = 1$

$$\mathbf{V_0}(n) = \left[\frac{2(1 + jn)}{2 + 3jn}\right] \mathbf{V_s}(n)$$

Since $\mathbf{V_s}(dc) = \frac{1}{2}$

$$\mathbf{V_0}(dc) = \left(\frac{2}{2}\right)\left(\frac{1}{2}\right)$$

$$= \frac{1}{2}$$

Furthermore,

$$\mathbf{V_0}(\omega_0) = \frac{-4}{\pi^2}\left[\frac{2(1 + j)}{2 + 3j}\right] = 0.318 \angle 168.69°$$

$$\mathbf{V_0}(3\omega_0) = \frac{-4}{9\pi^2}\left[\frac{2(1 + j3)}{2 + j9}\right] = 3.09 \times 10^{-2} \angle 174.09°$$

$$\mathbf{V_0}(5\omega_0) = \frac{-4}{25\pi^2}\left[\frac{2(1 + j5)}{2 + j15}\right] = 1.09 \times 10^{-2} \angle 176.28°$$

$$\mathbf{V_0}(7\omega_0) = \frac{-4}{49\pi^2}\left[\frac{2(1 + j7)}{2 + j21}\right] = 5.54 \times 10^{-3} \angle 177.31°$$

Hence,

$$v_0(t) = \frac{1}{2} + 0.318 \cos(t + 168.69°) + 3.09 \times 10^{-2} \cos(3t + 174.09°)$$
$$+ 1.09 \times 10^{-2} \cos(5t + 176.28°) + 5.54 \times 10^{-3} \cos(7t + 177.31°) + \ldots$$

14.5 The input function to the network can be expressed in the form

$$\mathbf{V}_s(j\omega) = \frac{10}{j\omega + 2}$$

The transfer function for the network obtained in the previous problem is

$$\mathbf{G}(j\omega) = \frac{2(1 + j\omega)}{2 + 3j\omega}$$

Then using the time convolution property of the Fourier transform we can express the output of the circuit in the form

$$\begin{aligned}\mathbf{V}_0(j\omega) &= \mathbf{G}(j\omega)\,\mathbf{V}_s(j\omega) \\ &= \left[\frac{2(1 + j\omega)}{2 + 3j\omega}\right]\left[\frac{10}{2 + j\omega}\right] \\ &= \frac{\frac{20}{3}(j\omega + 1)}{(j\omega + 2)\left(j\omega + \frac{2}{3}\right)}\end{aligned}$$

which can be written as a partial fraction expansion of the form

$$\frac{\frac{20}{3}(j\omega + 1)}{(j\omega + 2)\left(j\omega + \frac{2}{3}\right)} = \frac{A}{j\omega + 2} + \frac{B}{j\omega + \frac{2}{3}}$$

Evaluating the constants yields

$$\left.\frac{\frac{20}{3}(j\omega + 1)}{\left(j\omega + \frac{2}{3}\right)}\right|_{j\omega = -2} = A = 5$$

$$\left.\frac{\frac{20}{3}(j\omega + 1)}{(j\omega + 2)}\right|_{j\omega = -\frac{2}{3}} = B = \frac{5}{3}$$

Therefore,

$$\mathbf{V_0}(j\omega) = \frac{5}{j\omega + 2} + \frac{\frac{5}{3}}{j\omega + \frac{2}{3}}$$

And

$$v_0(t) = \left[5\, e^{-2t} + \frac{5}{3}\, e^{-\frac{2}{3}t}\right] u(t)\ \text{V}$$